The moduli of representations with Borel mold

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The moduli of representations is very useful for studying representations and for describing the moduli spaces of various geometric objects. There are several styles for constructing the moduli of representations. One way is taking the quotient of the whole of the representation variety in [9] and [4] by PGL. This method has a weak point: in [4] we constructed the coarse moduli scheme of equivalence classes of absolutely irreducible representations as the universal geometric quotient of the open subset consisting of stable points in the representation variety. However, on the complement of the absolutely irreducible representations, two representations which have the same composition factors become one point in the moduli of equivalence classes of representations. When two distinct representations have the same invariants, we can not separate them in the moduli. If we want to separate two distinct representations, we must choose another style.

In this paper we propose another style for constructing the moduli of representations in the non-absolutely irreducible case. We introduce the notion of "mold". A mold is, so to say, a subalgebra of the full matrix ring. We say that two representations have the same mold if their images generate the same type of subalgebras of the full matrix ring. By using the notion of mold, we collect representations which have the same mold, and we construct the moduli of representations with a fixed mold. As an example of molds, we consider a Borel mold, that is, the subalgebra of upper triangular matrices (up to inner automorphisms). The main purpose of this article is the construction of the moduli of (equivalence classes of) representations with Borel mold. This article is the first to develop "mold program", that is, the construction of moduli schemes of representations from a viewpoint of mold. In another papers, we will construct several moduli schemes of representations with various molds. In [5] we will deal with several molds of degree 2. In the degree 2 case, each molds over a field can be classified into 6 types. The moduli of representation with Borel mold is one of 6 types of the moduli of representations. In [6], we have calculated the cohomology ring of the moduli of representations with Borel mold for free monoids.

It is interesting and important to investigate the moduli of representations with Borel mold for several groups and monoids. Let us give an interesting example here. Let $\mathrm{Rep}_n(\Gamma)_B$ be the representation variety

with Borel mold, that is, the subscheme consisting of representations with Borel mold in the representation variety. Let us consider the universal representation with Borel mold on $\mathrm{Rep}_n(\Gamma)_B$. The universal representation induces the action on the trivial vector bundle $\mathcal{O}^{\oplus n}_{\mathrm{Rep}_n(\Gamma)_B}$ on $\mathrm{Rep}_n(\Gamma)_B$. Then there exists a unique filtration of Γ-invariant subbundles $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{O}^{\oplus n}_{\mathrm{Rep}_n(\Gamma)_B}$ with $\mathrm{rk}\ \mathcal{E}_i = i$. When $n = 2$ and Γ is a free monoid (or a free group) of rank 2, the universal sub-line bundle $\mathcal{E}_1$ is not trivial on $\mathrm{Rep}_2(\Gamma)_B$, however $\mathcal{E}_1^{\otimes 2}$ is trivial. From this fact, we see that each 2-dimensional representation with Borel mold of a group generated by two elements on $\mathrm{Spec}R$ with $(\mathrm{Pic}\,(\mathrm{Spec}R))_2 = 0$ can be normalized into a representation in upper triangular matrices (Corollary 4.9). Here we denote by $(\mathrm{Pic}\,(\mathrm{Spec}R))_2$ the 2-torsion part of the Picard group $\mathrm{Pic}\,(\mathrm{Spec}R)$. This fact shows one of geometric aspects of representations on schemes.

By “global representation theory” we understand a theory of representations on schemes, while by “local representation theory” we understand a theory of representations on fields or local rings. ”Global representation theory” has several geometric aspects like Corollary 4.9. The authors hopes that this article contributes to development of “global representation theory”.

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Several results of this article have been used in [6], [7], [8] and so on. It is not good to leave this article unpublished or not open, and hence the author submitted it to arXiv.org.

1. MOLD

In this section, we introduce the notion of *mold*. This notion is used for classification of representations and for constructing the moduli of representations. We collect representations which have the same mold, and we attach a canonical scheme structure on the collection. From a viewpoint of invariant theory, it is natural to classify representations with respect to mold.

Definition 1.1. Let X be a scheme. A subsheaf of $\mathcal{O}_X$-algebras $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ is said to be a *mold* of degree n on X if $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves on X. For a commutative ring R, we say that an R-subalgebra $A \subseteq \mathrm{M}_n(R)$ is a *mold* of degree n over R if A is a mold of degree n on $\mathrm{Spec}\,R$.

We introduce the moduli of molds, that is, the moduli of subalgebras of the full matrix ring as follows:

Definition and Proposition 1.2. *The following contravariant functor is representable by a closed subscheme of the Grassmann scheme.*

$$\begin{array}{rcccl} \mathcal{M}old_{n,d} & : & (\mathbf{Sch}) & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\mathcal{A} \mid \ \textit{a mold of deg } n \textit{ on } X \textit{ with } \mathrm{rk}\mathcal{A} = d\}. \end{array}$$

We denote by $\mathrm{Mold}_{n,d}$ *the scheme representing the functor* $\mathcal{M}old_{n,d}$.

Proof. Let $\mathrm{Grass}(d, \mathrm{M}_n)$ be the Grassmann scheme of rank d subbundles of M_n. The condition that a subbundle $\mathcal{A} \subset \mathrm{M}_n$ is closed under the multiplication of M_n and that $\mathcal{A}$ has the identity matrix is a closed condition. Hence the functor $\mathcal{M}old_{n,d}$ is representable by the closed subscheme of $\mathrm{Grass}(d, \mathrm{M}_n)$ defined by the condition above. □

We give some examples of the moduli of molds.

Example 1.3. In the case $n = 2$, we have

$$\mathrm{Mold}_{2,1} = \mathrm{Spec}\,\mathbb{Z}, \tag{1}$$

$$\mathrm{Mold}_{2,2} = \mathbb{P}^2_{\mathbb{Z}}, \tag{2}$$

$$\mathrm{Mold}_{2,3} = \mathbb{P}^1_{\mathbb{Z}}, \tag{3}$$

$$\mathrm{Mold}_{2,4} = \mathrm{Spec}\,\mathbb{Z}. \tag{4}$$

Indeed, (1) and (4) are obvious. To see (2), note that giving an R-valued point of $\mathrm{Mold}_{2,2}$ is equivalent to giving a rank 1 projective submodule of $\mathrm{M}_2(R)/R \cdot I_2$ for each commutative ring R. Hence we have $\mathrm{Mold}_{2,2} = \mathbb{P}^2_{\mathbb{Z}}$. Later (3) will be proved in Corollary 1.17.

We introduce an equivalence relation among molds as follows.

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be molds of degree n on a scheme X. We say that $\mathcal{A}$ and $\mathcal{B}$ are *locally equivalent* if for each $x \in X$ there exist an neighborhood U of x and $P_x \in \mathrm{GL}_n(\mathcal{O}_U)$ such that $P_x^{-1}(\mathcal{A}\mid_U)P_x = \mathcal{B}\mid_U \subseteq \mathrm{M}_n(\mathcal{O}_U)$.

We define the following typical molds, a Borel mold and a parabolic mold.

Definition 1.5. We define the mold $\mathcal{B}_n$ of degree n on $\mathrm{Spec}\,\mathbb{Z}$ by

$$\mathcal{B}_n := \{(b_{ij}) \in \mathrm{M}_n(\mathbb{Z}) \mid b_{ij} = 0 \text{ for } i > j\}.$$

Let $\mathcal{A}$ be a mold of degree n on a scheme X. We say that $\mathcal{A}$ is a *Borel mold* of degree n if $\mathcal{A}$ and $\mathcal{B}_n \otimes_{\mathbb{Z}} \mathcal{O}_X$ are locally equivalent.

Definition 1.6. Let $n_1, n_2, \dots, n_r$ be positive integers with $\sum n_i = n$. We define the mold $\mathcal{P}_{n_1,n_2,\dots,n_r}$ of degree n on $\operatorname{Spec}\mathbb{Z}$ by

$$\mathcal{P}_{n_1,n_2,\dots,n_r} := \left\{ (b_{ij}) \in \mathrm{M}_n(\mathbb{Z}) \;\middle|\; \begin{array}{c} b_{ij} = 0 \text{ if } \sum_{k=1}^{s} n_k < i \leq \sum_{k=1}^{s+1} n_k \\ \text{and } j \leq \sum_{k=1}^{s} n_k \end{array} \right\}.$$

Let $\mathcal{A}$ be a mold of degree n on a scheme X. We say that $\mathcal{A}$ is a *parabolic mold* of type $(n_1, n_2, \dots, n_r)$ if $\mathcal{A}$ and $\mathcal{P}_{n_1,n_2,\dots,n_r} \otimes_{\mathbb{Z}} \mathcal{O}_X$ are locally equivalent.

Let us discuss the structure of the moduli of molds $\mathcal{M}old_{n,d}$. The following case is easy.

Proposition 1.7. *For a positive integer n, we have*

$$\begin{aligned} \mathrm{Mold}_{n,n^2} &= \operatorname{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{n,d} &= \emptyset \qquad\qquad \textit{if} \quad n^2 - n + 1 < d < n^2. \end{aligned}$$

Proof. Since there is no rank n^2 mold of degree n except M_n, we have $\mathrm{Mold}_{n,n^2} = \operatorname{Spec}\mathbb{Z}$. Suppose that $n^2 - n + 1 < d < n^2$ and that $A \subseteq \mathrm{M}_n(k)$ is a rank d mold over an algebraically closed field k. Then A has a non-trivial invariant subspace of k^n, and hence A has at most dimension $n^2 - n + 1$. This is a contradiction. Because there exists no geometric point of $\mathrm{Mold}_{n,d}$ if $n^2 - n + 1 < d < n^2$, we obtain $\mathrm{Mold}_{n,d} = \emptyset$. □

Let $n_1, n_2, \dots, n_r$ be positive integers. Put $n := \sum_{1 \leq i \leq r} n_i$ and $d := \sum_{1 \leq i < j \leq r} n_i n_j$. We show that the moduli of molds $\mathrm{Mold}_{n,d}$ contains an open and closed subscheme corresponding to the parabolic molds of type $(n_1, n_2, \dots, n_r)$. This subscheme is isomorphic to a flag scheme, and hence it it smooth over $\mathbb{Z}$. To prove this statement, we make several preparations.

Notation 1.8. Let $n_1, n_2, \dots, n_r$ be positive integers.
Put $n := \sum n_i$. We define the closed subgroup scheme $P_{n_1,n_2,\dots,n_r}$ of PGL_n by

$$P_{n_1,n_2,\dots,n_r} := \left\{ (b_{ij}) \in \mathrm{PGL}_n \;\middle|\; \begin{array}{c} b_{ij} = 0 \quad \text{if } \sum_{\ell=1}^{s} n_\ell < i \leq \sum_{\ell=1}^{s+1} n_\ell \\ \text{and } j \leq \sum_{\ell=1}^{s} n_\ell \end{array} \right\}.$$

We denote by $\mathrm{Flag}_{n_1,n_2,\dots,n_r}$ the flag scheme $\mathrm{PGL}_n / P_{n_1,n_2,\dots,n_r}$.

Lemma 1.9. *Let R be a local ring. Let us consider the canonical action of the parabolic mold $\mathcal{P}_{n_1,n_2,\dots,n_r} \otimes_{\mathbb{Z}} R$ on R^n with $n = \sum n_i$. Then for each $1 \leq s \leq r$ there exists a unique rank $n_1 + n_2 + \cdots + n_s$ subbundle of R^n which is invariant under the parabolic mold.* (*By a subbundle M*

of R^n we understand an R-projective module M of R^n such that R^n/M is also projective.)

Proof. It is obvious that there exists an invariant rank $n_1 + \cdots + n_s$ subbundle of R^n. For proving the uniqueness, we only have to show that the Borel mold $\mathcal{B}_n \otimes_{\mathbb{Z}} R$ has a unique invariant rank i subbundle of R^n for each $1 \leq i \leq n$. Let $\{e_1, e_2, \ldots, e_n\}$ be the canonical basis of R^n. Suppose that $M \subseteq R^n$ is an invariant rank i subbundle. Then we show that $M = Re_1 + Re_2 + \cdots Re_i$. If $v = \sum a_j e_j \in M$ with $a_j \in R^\times$ and $j > i$, then $\{E_{1j}v, E_{2j}v, \ldots, E_{jj}v\}$ spans a rank j subbundle of M. This is a contradiction. Hence if $v = \sum a_j e_j \in M$, then $a_j \in m$ for each $j > i$, where m is a unique maximal ideal of R.

Let us define the projection $p : R^n \to R^i$ by $\sum a_j e_j \mapsto a_1 e_1 + \cdots + a_i e_i$. Since M is a rank i subbundle of R^n, $p|_M \colon M \to R^i$ is an isomorphism. For $1 \leq j \leq i$, we put $x_j := (p|_M)^{-1}(e_j)$.

We can write $x_j = e_j + v_j$ with $v_j \in me_{i+1} + \cdots + me_n$. Then we have $E_{jj}x_j = e_j = (p|_M)^{-1}(e_j) = x_j = e_j + v_j$, which implies that $v_j = 0$. Therefore $M = Re_1 + Re_2 + \cdots Re_i$. □

Corollary 1.10. *Let R be a local ring. Then the set $\mathcal{N}(\mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R) := \{Q \in \mathrm{PGL}_n(R) \mid Q \cdot (\mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R) \cdot Q^{-1} \subseteq \mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R\}$ is equal to $P_{n_1,\ldots,n_r}(R)$.*

Proof. Let $Q \in \mathcal{N}(\mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R)$. Take a representative of Q in $\mathrm{GL}_n(R)$, say it Q, too. Since the parabolic mold $\mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R = Q \cdot (\mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R) \cdot Q^{-1}$ has a unique invariant rank $n_1 + \cdots + n_s$ subbundles of R^n, Q also leaves such a subbundle invariant. Hence $Q \in P_{n_1,\ldots,n_r}(R)$. □

Proposition 1.11. *Let R be a local ring. For $Q \in P_{n_1,\ldots,n_r}(R)$, we define the algebra homomorphism $\mathrm{Ad}(Q) : \mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R \to \mathcal{P}_{n_1,\ldots,n_r} \otimes_{\mathbb{Z}} R$ by $Ad(Q)(X) = QXQ^{-1}$. If $\mathrm{Ad}(Q) = id$, then $Q = I_n$ in $P_{n_1,\ldots,n_r}(R)$.*

Proof. Let $Q = (q_{ij}) \in P_{n_1,\ldots,n_r}(R)$. Suppose that $\mathrm{Ad}(Q) = id$. Let us consider the block $Q_k := (q_{ij})_{\sum_{m=1}^{k-1} n_m < i,j \leq \sum_{m=1}^{k} n_m}$ for $1 \leq k \leq r$. By the hypothesis, $\mathrm{Ad}(Q_k) : \mathrm{M}_{n_k}(R) \to \mathrm{M}_{n_k}(R)$ is the identity for $1 \leq k \leq r$. Then we see that Q_k is a scalar matrix. For $i < j$, considering the (i,j)-entry of $QE_{ij}Q^{-1} = E_{ij}$, we have $q_{ii}/q_{jj} = 1$. Hence $q_{11} = q_{22} = \cdots = q_{nn}$. For $i < j$, considering the (i,j)-entry of $QE_{jj}Q^{-1} = E_{jj}$, we have $q_{ij}/q_{jj} = 0$. Hence we obtain $q_{ij} = 0$. Therefore $Q = I_n$ in $P_{n_1,\ldots,n_r}(R)$. □

We construct a closed subscheme of the moduli of molds in the next proposition.

Proposition 1.12. *Let $n_1, n_2, \ldots, n_r$ be positive integers. Put $n := \sum n_i$ and $d := \sum_{1 \le i \le j \le n} n_i n_j$. We define $\phi : \mathrm{PGL}_n \to \mathrm{Mold}_{n,d}$ by $Q \mapsto Q(\mathcal{P}_{n_1,n_2,\ldots,n_r} \otimes_{\mathbb{Z}} \mathcal{O}_X)Q^{-1}$ for a X-valued point Q of PGL_n with a scheme X. Then the morphism ϕ induces the closed immersion $\mathrm{Flag}_{n_1,n_2,\ldots,n_r} \to \mathrm{Mold}_{n,d}$. As a closed subscheme $\mathrm{Flag}_{n_1,n_2,\ldots,n_r}$ corresponds to the parabolic molds of type $(n_1, n_2, \ldots, n_r)$.*

Proof. The morphism ϕ induces $\overline{\phi} : \mathrm{PGL}_n/P_{n_1,\ldots,n_r} = \mathrm{Flag}_{n_1,\ldots,n_r} \to \mathrm{Mold}_{n,d}$. We claim that $\overline{\phi}$ is a closed immersion. First we show that $\overline{\phi}$ is a monomorphism. Let X be a scheme. Let P and Q be X-valued points of PGL_n. Suppose that $\phi(P)$ and $\phi(Q)$ are same molds on X. Since $P(\mathcal{P}_{n_1,\ldots,n_r} \otimes \mathcal{O}_{X,x})P^{-1} = Q(\mathcal{P}_{n_1,\ldots,n_r} \otimes \mathcal{O}_{X,x})Q^{-1}$ for each $x \in X$, $P^{-1}Q$ is contained in the normalizer $N(\mathcal{P}_{n_1,\ldots,n_r} \otimes \mathcal{O}_{X,x})$ of $\mathcal{P}_{n_1,\ldots,n_r} \otimes \mathcal{O}_{X,x}$. From Corollary 1.10 we see that $P^{-1}Q \in \mathrm{P}_{n_1,\ldots,n_r}$ at each x and hence that $P = Q$ in $\mathrm{PGL}_n/\mathrm{P}_{n_1,\ldots,n_r}$. Therefore $\overline{\phi}$ is a monomorphism. Next the morphism $\overline{\phi}$ is proper, since the scheme $\mathrm{PGL}_n/\mathrm{P}_{n_1,\ldots,n_r}$ is proper over $\mathbb{Z}$. Thus we have proved that $\overline{\phi}$ is a closed immersion. □

The closed subscheme constructed above is also open in the moduli of molds. For proving this, we introduce the following propositions.

Lemma 1.13. *Let R be a local ring and let $A \subseteq \mathrm{M}_n(R)$ be a parabolic mold over R. Then the normalizer $N(A) := \{X \in \mathrm{M}_n(R) \mid [X, Y] \in A \text{ for each } Y \in A\}$ is equal to A. Here we define $[X, Y] := XY - YX$.*

Proof. Since R is a local ring, by changing A to PAP^{-1} with a suitable matrix $P \in \mathrm{GL}_n(R)$ we may assume that

$$(5) A = \left\{ (b_{ij}) \in \mathrm{M}_n(R) \;\middle|\; \begin{array}{cc} b_{ij} = 0 & \text{if } \sum_{\ell=1}^{s} n_\ell < i \le \sum_{\ell=1}^{s+1} n_\ell \\ & \text{and } j \le \sum_{\ell=1}^{s} n_\ell \end{array} \right\}.$$

It is clear that $A \subseteq N(A)$. Suppose that $X = \sum a_{ij}E_{ij} \in \mathrm{M}_n(R) \setminus A$. There exists $E_{ij} \notin A$ with $a_{ij} \neq 0$. Note that $i > j$. For $E_{jj} \in A$, we have $[X, E_{jj}] = a_{ij}E_{ij} + \cdots$. Since the (i,j)-entry of $[X, E_{ij}]$ is not zero, $[X, E_{jj}] \notin A$. Hence $X \notin N(A)$. Thus we have proved that $A = N(A)$. □

Proposition 1.14. *Let k be a field and let $A \subseteq \mathrm{M}_n(k)$ be a parabolic mold over k. Then the linear map*

$$\begin{array}{ccc} \mathrm{M}_n(k)/A & \to & \mathrm{Der}_k(A, \mathrm{M}_n(k)/A) \\ X & \mapsto & [X, -] \end{array}$$

is bijective.

Proof. We can easily check that the above map is well-defined. The injectivity of the map follows from Lemma 1.13. For proving that the linear map is an isomorphism, we may assume (5). Let $\delta \in \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$. If $E_{ij} \in A$, then we have

$$\delta(E_{ij}) = \delta(E_{ii}E_{ij}) = E_{ii}\delta(E_{ij}) + \delta(E_{ii})E_{ij}$$

and

$$\delta(E_{ij}) = \delta(E_{ij}E_{jj}) = E_{ij}\delta(E_{jj}) + \delta(E_{ij})E_{jj}.$$

The first equality shows that the $(\ell, *)$-entries of $\delta(E_{ij})$ are determined by $\delta(E_{ii})$ for $\ell \neq i$, and the second equality shows that the $(*, \ell)$-entries of $\delta(E_{ij})$ are determined by $\delta(E_{jj})$ for $\ell \neq j$. Hence $\delta(E_{ij})$ is determined by $\delta(E_{ii})$ and $\delta(E_{jj})$. The derivation δ is determined by $\{\delta(E_{ii}) \mid 1 \leq i \leq n\}$.

Since $\delta(E_{ii}) = \delta(E_{ii}E_{ii}) = E_{ii}\delta(E_{ii}) + \delta(E_{ii})E_{ii}$, the matrix $\delta(E_{ii})$ has zero entries except $(i, *)$-entries and $(*, i)$-entries. For $i \neq j$, we obtain

$$0 = \delta(E_{ii}E_{jj}) = E_{ii}\delta(E_{jj}) + \delta(E_{ii})E_{jj}.$$

If $j \leq \sum_{\ell=1}^{s} n_\ell < i$ for some s, then $\delta(E_{ii})_{ij} = -\delta(E_{jj})_{ij}$. We see that δ is determined by the data $\{\delta(E_{jj})_{ij} \mid j \leq \sum_{\ell=1}^{s} n_\ell < i \text{ for some } s\}$ and that $\dim \mathrm{Der}_k(A, \mathrm{M}_n(k)/A) \leq \sum_{1 \leq i < j \leq r} n_i n_j = \dim \mathrm{M}_n(k)/A$. From the injectivity, we prove that the above linear map is an isomorphism. □

Now we prove that the closed subscheme $\mathrm{Flag}_{n_1,\dots,n_r}$ is open in $\mathrm{Mold}_{n,d}$.

Proposition 1.15. *The morphism* $\overline{\phi} : \mathrm{Flag}_{n_1,\dots,n_r} \to \mathrm{Mold}_{n,d}$ *in Proposition 1.12 is smooth. In particular,* $\mathrm{Flag}_{n_1,\dots,n_r}$ *is an open and closed subscheme of* $\mathrm{Mold}_{n,d}$.

Proof. Let (R, m) be an artin local ring. Let I be an ideal with $m \cdot I = 0$. Suppose that $A \subseteq \mathrm{M}_n(R)$ is a mold over R such that $A \otimes_R R/I$ is a parabolic mold of type $(n_1, \dots, n_r)$ over R/I. For proving the statement, we only have to show that A is a parabolic mold over R. From the assumption, there exists $\overline{P} \in \mathrm{GL}_n(R/I)$ such that $\overline{P}(A \otimes_R R/I)\overline{P}^{-1} = \mathcal{P}_{n_1,\dots,n_r} \otimes_{\mathbb{Z}} R/I$. Take a matrix $P \in \mathrm{GL}_n(R)$ such that $P \bmod m = \overline{P}$. By changing A to PAP^{-1}, we may assume that A is a mold over R such that $A \otimes_R R/I = \mathcal{P}_{n_1,\dots,n_r} \otimes_{\mathbb{Z}} R/I$. We denote $A \otimes_R R/I$ by $\overline{A}$.

Let $q : \mathrm{M}_n(R) \to \mathrm{M}_n(R)$ be the R-linear map defined by

$$q(X)_{ij} := \begin{cases} 0 & \text{if } \sum_{\ell=1}^{s} n_\ell < i \leq \sum_{\ell=1}^{s+1} n_\ell, j \leq \sum_{\ell=1}^{s} n_\ell \text{ for some } s \\ x_{ij} & \text{otherwise} \end{cases}$$

for $X = (x_{ij}) \in \mathrm{M}_n(R)$. For matrix elements $E_{ij} \in \overline{A}$ choose their representatives $\tilde{E}_{ij}$ in A. We define the R/I-linear map $\delta : \overline{A} \to \mathrm{M}_n(I)$ by $\delta(\sum a_{ij}E_{ij}) := q(\sum \tilde{a}_{ij}\tilde{E}_{ij})$, where $\tilde{a}_{ij} \in R$ is a representative of $a_{ij} \in R/I$. Since $I^2 = 0$, we can easily check that δ is independent of choices of $\tilde{a}_{ij}$.

Set $R/m = k$. Note that $\mathrm{M}_n(I) = \mathrm{M}_n(R) \otimes_R I \otimes_R R/m = \mathrm{M}_n(k) \otimes_k I$. The map δ induces the k-linear map $\overline{\delta} : A \otimes_R k \to \mathrm{M}_n(k) \otimes_k I$. From the definition of δ we see that $\overline{\delta}$ is a derivation. By Proposition 1.14 we have $Y \in \mathrm{M}_n(k) \otimes_k I = \mathrm{M}_n(I)$ such that $\overline{\delta} = [Y, -]$. Hence the map $A \overset{\text{proj.}}{\to} \overline{A} \overset{\delta}{\to} \mathrm{M}_n(I)$ is given by $X \mapsto [Y, X]$. Putting $P := (I_n - Y) \in \mathrm{GL}_n(R)$, we have $PXP^{-1} = (I_n - Y)X(I_n + Y) = X - [Y, X]$ for $X \in \mathrm{M}_n(R)$. We see that $PAP^{-1} = \mathcal{P}_{n_1,\dots,n_r} \otimes_{\mathbb{Z}} R$ and hence that A is a parabolic mold over R. This completes the proof. $\square$

From the discussion above, we obtain the following theorem.

Theorem 1.16. *Let $n_1, n_2, \dots, n_r$ be positive integers. Put $n := \sum n_i$ and $d := \sum_{1 \le i \le j \le n} n_i n_j$. Then the moduli of molds* $\mathrm{Mold}_{n,d}$ *contains the open and closed subscheme corresponding to the parabolic molds of type $(n_1, n_2, \dots, n_r)$. This subscheme is isomorphic to a flag scheme over $\mathbb{Z}$.*

The above theorem follows the next corollary.

Corollary 1.17. *In the case $d = n^2 - n + 1$, we have*

$$\mathrm{Mold}_{n,n^2-n+1} \cong \begin{cases} \mathbb{P}^{n-1}_{\mathbb{Z}} \coprod \mathbb{P}^{n-1}_{\mathbb{Z}} & n > 2 \\ \mathbb{P}^1_{\mathbb{Z}} & n = 2. \end{cases}$$

Proof. Each geometric point in $\mathrm{Mold}_{n,n^2-n+1}(\Omega)$, that is, each mold of rank $n^2 - n + 1$ over Ω has an invariant subspace of Ω^n. This mold is a parabolic mold of type $(1, n-1)$ or $(n-1, 1)$. Hence the moduli $\mathrm{Mold}_{n,n^2-n+1}$ is covered by $\mathrm{Flag}_{1,n-1}$ and $\mathrm{Flag}_{n-1,1}$. $\square$

From now on we prepare some terminologies on representations. Using the notion of mold, we classify representations.

Definition 1.18. Let Γ be a group or a monoid. Let X be a scheme. By a *representation* of degree n on X for Γ we understand a group homomorphism (resp. a monoid homomorphism) $\rho : \Gamma \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$ (resp. $\rho : \Gamma \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$), where $\Gamma(X, \mathcal{O}_X)$ is the ring of global sections on X.

For two representations ρ, ρ' of degree n for Γ on a scheme X, we say that ρ and ρ' are *equivalent* (or $\rho \sim \rho'$) if there exists a $\Gamma(X, \mathcal{O}_X)$-algebra isomorphism $\sigma : \mathrm{M}_n(\Gamma(X, \mathcal{O}_X)) \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$ such that

$\sigma(\rho(\gamma)) = \rho'(\gamma)$ for each $\gamma \in \Gamma$. We also say that ρ and ρ' are *locally equivalent* if there exists an open covering $X = \cup_{i\in I} U_i$ such that $\rho\,|_{U_i}$ and $\rho'\,|_{U_i}$ are equivalent for each $i \in I$.

Definition 1.19. Let $\mathcal{A}$ be a mold of degree n on X. For a representation ρ on X, we say that ρ is a *representation with mold* $\mathcal{A}$ if the subsheaf $\mathcal{O}_X[\rho(\Gamma)]$ of $\mathcal{O}_X$-albebras of $\mathrm{M}_n(\mathcal{O}_X)$ is locally equivalent to $\mathcal{A}$. In particular, we say that ρ is a *representation with Borel mold* if $\mathcal{O}_X[\rho(\Gamma)]$ is a Borel mold. We also say that ρ is a *representation with parabolic mold of type* $(n_1, n_2, \dots, n_r)$ if $\mathcal{O}_X[\rho(\Gamma)]$ is a parabolic mold of type $(n_1, n_2, \dots, n_r)$.

In [4] we proved the existence of the coarse moduli scheme of equivalence classes of absolutely irreducible representations. Here we quote this result.

Definition 1.20. For a representation ρ of degree n for a group Γ on a scheme X, we say that ρ is *absolutely irreducible* if $\mathcal{O}_X[\rho(\Gamma)] = \mathrm{M}_n(\mathcal{O}_X)$. This definition is equivalent to the one in [4]. We abbreviate an absolutely irreducible representation to a.i.r.

Theorem 1.21 ([4])**.** *There exists a coarse moduli scheme separated over $\mathbb{Z}$ associated to the following moduli functor:*

$$\begin{array}{cccll} \mathcal{E}q\mathcal{A}IR_n(\Gamma) & : & (\mathbf{Sch}) & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\rho : \textit{ a.i.r. of degree } n \textit{ for } \Gamma \textit{ on } X\}/\sim . \end{array}$$

In particular, if Γ is a finitely generated group (or monoid), then the moduli is of finite type over $\mathbb{Z}$.

In the sequel, we only deal with representations with Borel mold. We will construct the moduli schemes of representations with Borel mold.

2. Construction of the moduli of representations with Borel mold

In this section, we construct the moduli scheme of equivalence classes of representations with Borel mold.

Let us recall the representation variety. Let Γ be a group or a monoid. The following contravariant functor is representable by an affine scheme:

$$\begin{array}{cccll} \mathrm{Rep}_n(\Gamma) & : & (\mathbf{Sch}) & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\rho : \text{ rep. of deg } n \text{ for } \Gamma \text{ on } X\ \}. \end{array}$$

We call the affine scheme $\mathrm{Rep}_n(\Gamma)$ the representation variety of degree n for Γ. The group scheme PGL_n over $\mathbb{Z}$ acts on $\mathrm{Rep}_n(\Gamma)$ by $\rho \mapsto P^{-1}\rho P$. Each PGL_n-orbit forms an equivalence classes of representations.

For a commutative ring R, we set $\mathcal{B}_n(R) := \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ for each } i > j\}$, that is, $\mathcal{B}_n(R)$ is the R-subalgebra of upper triangular matrices. We define the closed subgroup scheme B_n of PGL_n by $\mathrm{B}_n := \{(a_{ij}) \in \mathrm{PGL}_n \mid a_{ij} = 0 \text{ for } i > j\}$. By a *$\mathcal{B}_n$-representation* of degree n for Γ on a scheme X, we understand a homomorphism $\rho : \Gamma \to \mathcal{B}_n(\Gamma(X, \mathcal{O}_X))$. It is easy to check that the subfunctor of $\mathrm{Rep}_n(\Gamma)$ consisting of $\mathcal{B}_n$-representations is represented by a closed subscheme. We denote it by $\mathrm{B}_n(\Gamma)$.

For two $\mathcal{B}_n$-representations ρ and ρ' of degree n for Γ on a scheme X, we say that ρ and ρ' are *B_n-equivalent* to each other, if there exists a X-valued point $Q \in \mathrm{B}_n(X)$ such that $Q\rho Q^{-1} = \rho'$. The group scheme B_n acts on $\mathrm{B}_n(\Gamma)$ by $\rho \mapsto Q\rho Q^{-1}$. Each B_n-orbit forms a B_n-equivalence classes of $\mathcal{B}_n$-representations.

By a *$\mathcal{B}_n$-representation with Borel mold* for Γ on a scheme X, we understand a homomorphism $\rho : \Gamma \to \mathcal{B}_n(\Gamma(X, \mathcal{O}_X))$ which is a representation with Borel mold. Note that $\rho : \Gamma \to \mathcal{B}_n(\Gamma(X, \mathcal{O}_X))$ is a $\mathcal{B}_n$-representation with Borel mold if and only if $\rho(\Gamma)$ generates $\mathcal{B}_n(\Gamma(X, \mathcal{O}_X))$ as a $\Gamma(X, \mathcal{O}_X)$-algebra. We denote by $\mathrm{Rep}_n(\Gamma)_B$ the subfunctor of $\mathrm{Rep}_n(\Gamma)$ consisting of representations with Borel mold. We also denote by $\mathrm{B}_n(\Gamma)_B$ the subfunctor of $\mathrm{B}_n(\Gamma)$ consisting of $\mathcal{B}_n$-representations with Borel mold. We see that $\mathrm{Rep}_n(\Gamma)_B$ and $\mathrm{B}_n(\Gamma)_B$ are subschemes of $\mathrm{Rep}_n(\Gamma)$ and $\mathrm{B}_n(\Gamma)$, respectively. Indeed, we can verify that $\mathrm{Rep}_n(\Gamma)$ has a (locally closed) subscheme $\mathrm{Rep}_n(\Gamma)_d$ of representations with mold of rank $d = n(n+1)/2$. Then the subscheme $\mathrm{Rep}_n(\Gamma)_B$ can be obtained by taking the pull-back of $\mathrm{Flag}_{1,1,\dots,1}$ by the morphism $\mathrm{Rep}_n(\Gamma)_d \to \mathrm{Mold}_{n,d}$. Similarly, we can verify that $\mathrm{B}_n(\Gamma)_B$ is an open subscheme of $\mathrm{B}_n(\Gamma)$. We call the scheme $\mathrm{Rep}_n(\Gamma)_B$ the *representation variety with Borel mold* of degree n for Γ.

Now let us define the contravariant functor $\mathcal{E}qB_n(\Gamma)$. By a *generalized representation with Borel mold* of degree n for Γ on a scheme X, we understand pairs $\{(U_i, \rho_i)\}_{i \in I}$ of an open set U_i and a representation with Borel mold $\rho_i : \Gamma \to \mathrm{M}_n(\Gamma(U_i, \mathcal{O}_X))$ satisfying the following two conditions:

(i) $\cup_{i \in I} U_i = X$,

(ii) for each $x \in U_i \cap U_j$, ρ_i and ρ_j are equivalent to each other on a neighbourhood of x.

We say that generalized representations with Borel mold $\{(U_i, \rho_i)\}_{i \in I}$ and $\{(V_j, \sigma_j)\}_{j \in J}$ are *equivalent* if $\{(U_i, \rho_i)\}_{i \in I} \cup \{(V_j, \sigma_j)\}_{j \in J}$ is a generalized representation again.

We introduce the contravariant functor $\mathcal{E}qB_n(\Gamma)$:

$$\begin{array}{rcccl} \mathcal{E}qB_n(\Gamma) & : & (\mathbf{Sch}) & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \{(U_i,\rho_i)\}_{i\in I} \;\middle|\; \begin{array}{l} \text{gen. rep. with} \\ \text{Borel mold of} \\ \text{deg } n \text{ for } \Gamma \text{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

In this section, we show that the functor $\mathcal{E}qB_n(\Gamma)$ is representable by a scheme over $\mathbb{Z}$. For proving this, we prepare another functor $\mathcal{E}qBB_n(\Gamma)$.

By a *generalized* $\mathcal{B}_n$*-representation* with Borel mold of degree n for Γ on a scheme X, we understand pairs $\{(U_i,\rho_i)\}_{i\in I}$ of an open set U_i and a $\mathcal{B}_n$-representation with Borel mold $\rho_i : \Gamma \to \mathcal{B}_n(\Gamma(U_i,\mathcal{O}_X))$ satisfying the above condition (i) and the following:

(ii)* for each $x \in U_i \cap U_j$, there exists $Q \in \mathrm{B}_n(V)$ such that $Q^{-1}\rho_i Q = \rho_j$ on a neighbourhood V of x.

We say that two generalized $\mathcal{B}_n$-representations $\{(U_i,\rho_i)\}_{i\in I}$ and $\{(V_j,\sigma_j)\}_{j\in J}$ are B_n*-equivalent* (or $\{(U_i,\rho_i)\}_{i\in I} \sim_{\mathrm{B}_n} \{(V_j,\sigma_j)\}_{j\in J}$) if $\{(U_i,\rho_i)\}_{i\in I} \cup \{(V_j,\sigma_j)\}_{j\in J}$ is a generalized $\mathcal{B}_n$-representation again.

We define the contravariant functor $\mathcal{E}qBB_n(\Gamma)$ by

$$\begin{array}{rcccl} \mathcal{E}qBB_n(\Gamma) & : & (\mathbf{Sch}) & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \{(U_i,\rho_i)\}_{i\in I} \;\middle|\; \begin{array}{l} \text{gen. } \mathcal{B}_n\text{-rep. with} \\ \text{Borel mold of} \\ \text{deg } n \text{ for } \Gamma \text{ on } X \end{array} \right\} \Big/ \sim_{\mathrm{B}_n} . \end{array}$$

We can check that there exists a canonical isomorphism $\mathcal{E}qBB_n(\Gamma) \to \mathcal{E}qB_n(\Gamma)$. Hence the representability of $\mathcal{E}qB_n(\Gamma)$ is reduced to the one of $\mathcal{E}qBB_n(\Gamma)$.

The following lemma can be easily verified:

Lemma 2.1. *The functor $\mathcal{E}qBB_n(\Gamma)$ is a sheaf with respect to Zariski topology.*

By the above lemma, for proving that $\mathcal{E}qBB_n(\Gamma)$ is representable, it suffices to show that it admits an open covering of affine schemes.

Let us consider the index set $\mathcal{I}_n := \{(i,j) \mid 1 \leq i \leq j \leq n\}$. We define the order on $\mathcal{I}_n$ by

$$(i,j) \leq (i',j') \Leftrightarrow \begin{cases} |i-j| < |i'-j'| \\ \text{or} \\ |i-j| = |i'-j'| \text{ and } i \leq i'. \end{cases}$$

We define $J_{i,j}(R) := \{(a_{kl}) \in \mathcal{B}_n(R) \mid a_{kl} = 0 \text{ for each } (k,l) \leq (i,j)\}$.

Let ρ be a $\mathcal{B}_n$-representation with Borel mold of degree n for Γ on a scheme X. We say that ρ satisfies the $(*)$-condition with respect to an $\mathcal{I}_n$-indexed subset $\{\alpha_{ij}\}_{(i,j)\in\mathcal{I}_n}$ of Γ if the set $\{\rho(\alpha_{k,\ell})\}_{(k,\ell)\leq(i,j)}$ forms a basis of $\mathcal{B}_n(\Gamma(X,\mathcal{O}_X))/J_{i,j}(\Gamma(X,\mathcal{O}_X))$ over $\Gamma(X,\mathcal{O}_X)$ for each $(i,j) \in \mathcal{I}_n$. For any $P \in \mathrm{B}_n(X)$, ρ satisfies the $(*)$-condition with respect to $\{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}_n}$ if and only if so does $P\rho P^{-1}$. We also say that a generalized $\mathcal{B}_n$-representation with Borel mold $\{(U_i,\rho_i)\}_{i\in I}$ *satisfies the* $(*)$*-condition* with respect to $\{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}_n}$ if the same condition holds.

For an $\mathcal{I}_n$-indexed set $A = \{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}_n}$, the subfunctor $\mathcal{E}q\underline{BB_n}(\Gamma)_A$ of $\mathcal{E}q\underline{BB_n}(\Gamma)$ is defined by $\mathcal{E}q\underline{BB_n}(\Gamma)_A(X) := \{\rho \in \mathcal{E}q\underline{BB_n}(\Gamma)(X) \mid \rho \text{ satisfies the } (*)\text{-condition with respect to } A\}$ for a scheme X. Let k be a field and let $\rho \in \mathcal{E}q\underline{BB_n}(\Gamma)(k)$. Let $\tilde{\rho} : \Gamma \to \mathcal{B}_n(k)$ be a representative of ρ. Then it is easy to check that ρ satisfies the $(*)$-condition with respect to some $\mathcal{I}_n$-indexed subset $A = \{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}_n}$ of Γ. Hence we have $\mathcal{E}q\underline{BB_n}(\Gamma)(k) = \cup_A \mathcal{E}q\underline{BB_n}(\Gamma)_A(k)$, where the union runs the $\mathcal{I}_n$-indexed subsets of Γ.

In the sequel, we show that $\mathcal{E}q\underline{BB_n}(\Gamma)_A$ is an affine scheme for each $\mathcal{I}_n$-indexed subset A of Γ. Let us define the subfunctor $\mathrm{B}_n(\Gamma)_{B,A}$ of $\mathrm{B}_n(\Gamma)$ by

$$\mathrm{B}_n(\Gamma)_{B,A}(X) := \{\sigma \in \mathrm{B}_n(\Gamma)(X) \mid \sigma \text{ satisfies the } (*)\text{-condition for } A\}$$

for a scheme X. We easily verify that $\mathrm{B}_n(\Gamma)_{B,A}$ is an affine open subscheme of $\mathrm{B}_n(\Gamma)$. The action of B_n on $\mathrm{B}_n(\Gamma)$ by $\rho \mapsto Q\rho Q^{-1}$ induces the one of B_n on $\mathrm{B}_n(\Gamma)_{B,A}$. There exists a canonical morphism $\mathrm{B}_n(\Gamma)_{B,A} \to \mathcal{E}q\underline{BB_n}(\Gamma)_A$. Then we obtain the following:

Lemma 2.2. *The morphism* $\mathrm{B}_n(\Gamma)_{B,A} \to \mathcal{E}q\underline{BB_n}(\Gamma)_A$ *is a* B_n*-principal fiber bundle. In particular, the functor* $\mathcal{E}q\underline{BB_n}(\Gamma)_A$ *is an affine scheme.*

As a corollary of Lemma 2.2, we have:

Corollary 2.3. *The functor* $\mathcal{E}q\underline{BB_n}(\Gamma)$ *is representable.*

Proof. The statement follows from that the sheaf $\mathcal{E}q\underline{BB_n}(\Gamma)$ is covered by affine schemes $\mathcal{E}q\underline{BB_n}(\Gamma)_A$. $\square$

Before proving Lemma 2.2, we need several preparations and long discussions. Let ρ be the universal $\mathcal{B}_n$-representation on $\mathrm{B}_n(\Gamma)$. Fix an $\mathcal{I}_n$-indexed subset $A = \{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}_n}$ of Γ. We denote the coordinate ring of the affine scheme $\mathrm{B}_n(\Gamma)_{B,A}$ by R. We define $\eta_{ij}(\gamma) \in J_{ij}(R)$ and $\varepsilon_{ij}(\gamma) \in R$ for $\gamma \in \Gamma$ and $(i,j) \in \mathcal{I}_n$ by induction. First we define $\varepsilon_{11}(\gamma) := \rho(\gamma)_{11}/\rho(\alpha_{11})_{11}$ and $\eta_{11}(\gamma) := \rho(\gamma) - \varepsilon_{11}(\gamma)\rho(\alpha_{11})$. Suppose that $\eta_{i'j'}(\gamma) \in J_{i'j'}(R)$ and $\varepsilon_{i'j'}(\gamma) \in R$ are defined for each $\gamma \in \Gamma$ and for each $(i',j') < (i,j)$. Now we define $\varepsilon_{ij}(\gamma) := (\eta_{i'j'}(\gamma))_{ij}/(\eta_{i'j'}(\alpha_{ij}))_{ij}$ and $\eta_{ij}(\gamma) := \eta_{i'j'}(\gamma) - \varepsilon_{ij}(\gamma)\eta_{i'j'}(\alpha_{ij})$, where (i',j') is the previous index of (i,j). Here we remark that $(\eta_{i',j'}(\alpha_{ij}))_{ij} \in R^\times$. Set $\tau_{11} := (\rho_{11}(\alpha_{11}))_{11} \in R^\times$ and $\tau_{ij} := (\eta_{i',j'}(\alpha_{ij}))_{ij} \in R^\times$ for $(i,j) \in \mathcal{I}_n \setminus \{(1,1)\}$.

Under the action of B_n on R, the elements τ_{ij} are semi-invariant. Indeed, for $Q = (b_{ij}) \in \mathrm{B}_n$ we have $Q^*\rho(\gamma) = Q\rho(\gamma)Q^{-1}$, $Q^*\eta_{ij}(\gamma) = Q\eta_{ij}(\gamma)Q^{-1}$, and $Q^*\tau_{ij} = (b_{ii}/b_{jj})\tau_{ij}$. Hence $\tau_{ij}\tau_{jk}\tau_{ik}^{-1}$ is B_n-invariant. We also see that $\varepsilon_{ij}(\gamma)$ is B_n-invariant.

Let us introduce the B_n-invariant subalgebra of R.

Definition 2.4. Let R^{ch} be the B_n-invariant subalgebra of R. We define the affine scheme $\mathrm{Ch}_n(\Gamma)_{B,A} := \mathrm{Spec} R^{\mathrm{ch}}$.

The next lemma is a key for proving Lemma 2.2:

Lemma 2.5. *There exists an upper triangular invertible matrix* $Q \in \tilde{B}_n(R) := \{(b_{ij}) \in \mathrm{GL}_n(R) \mid b_{ij} = 0$ *for* $i > j\}$ *such that all entries of* $Q\rho(\gamma)Q^{-1}$ *are contained in* R^{ch} *for each* $\gamma \in \Gamma$.

From now on, we concentrate ourselves into proving Lemma 2.5. Note that the representation ρ satisfies

$$\rho(\gamma) = \sum_{(i,j)\in\mathcal{I}_n} \varepsilon_{i,j}(\gamma)\eta_{i'j'}(\alpha_{ij})$$

for each $\gamma \in \Gamma$. Here we put $\eta_{1'1'}(\alpha_{11}) = \rho(\alpha_{11})$. For proving Lemma 2.5, we only need to show that there exists $Q \in \tilde{B}_n(R)$ such that $Q\eta_{i'j'}(\alpha_{ij})Q^{-1} \in \mathcal{B}_n(R^{\mathrm{ch}})$ for each $(i,j) \in \mathcal{I}_n$.

Definition 2.6. For $X \in \mathcal{B}_n(R)$, we say that X is *canonical* if the action of B_n on (entries of) X is described by $Q^*X = QXQ^{-1}$ for $Q \in \mathrm{B}_n$. Note that $\rho(\gamma)$ and $\eta_{ij}(\gamma)$ are canonical for each $\gamma \in \Gamma$ and $(i,j) \in \mathcal{I}_n$.

Definition 2.7. For $(i,j) \in \mathcal{I}_n$, we define the *convex hull* $\overline{(i,j)}$ of (i,j) as the subset $\overline{(i,j)} := \{(k,\ell) \in \mathcal{I}_n \mid k \le i,\ \ell \ge j\}$ of $\mathcal{I}_n$. We also define the *convex hull* $\overline{S}$ of a subset S of $\mathcal{I}_n$ by $\overline{S} := \cup_{(i,j)\in S} \overline{(i,j)}$.

Definition 2.8. Let $X = (x_{ij}) \in \mathcal{B}_n(R)$. We define the *support* of X by $\mathrm{Supp}X := \{(i,j) \in \mathcal{I}_n \mid x_{ij} \neq 0\}$. We say that X is (i,j)*-shaped* if $x_{ij} \in R^{\times}$ and $\mathrm{Supp}X \subseteq \overline{(i,j)}$. We also say that X is *well-shaped* if $X = 0$ or X is (i,j)-shaped for some $(i,j) \in \mathcal{I}_n$.

Set $Y(i,j) := \eta_{i'j'}(\alpha_{ij})$ for $(i,j) \in \mathcal{I}_n$. (Recall $Y(1,1) := \eta_{1'1'}(\alpha_{11}) = \rho(\alpha_{11})$.) Note that $Y(i,j)$ is canonical and that the (i,j)-entry of $Y(i,j)$ is contained in $R^{\times}$. Now we show the following lemma.

Lemma 2.9. *For each $(i,j) \in \mathcal{I}_n$, there exists an (i,j)-shaped canonical matrix $X(i,j) \in \mathcal{B}_n(R)$ such that*

$$Y(i,j) = X(i,j) + \sum_{(k,\ell) \in \overline{\mathrm{Supp}Y(i,j)} \setminus \overline{(i,j)}} a_{k,\ell}(i,j) X(k,\ell),$$

where $a_{k,\ell}(i,j)$ is B_n-invariant.

Proof. Since $Y(1,n)$ is a $(1,n)$-shaped canonical matrix, we set $X(1,n) := Y(1,n)$. Suppose that we can define a (k,ℓ)-shaped canonical matrix $X(k,\ell)$ for $(k,\ell) > (i,j)$ and that the equalities above hold. Let us consider the case (i,j). Set $J := \overline{\mathrm{Supp}Y(i,j)} \setminus \overline{(i,j)}$. If $J = \emptyset$, then set $X(i,j) := Y(i,j)$. Assume that $J \neq \emptyset$. Let (s,t) be the minimum element of J. Then $(s,t) \in \mathrm{Supp}Y(i,j)$. If $(s',t') \neq (s,t)$ for $(s',t') \in \mathrm{Supp}Y(i,j)$, then $(s,t) \notin \overline{(s',t')}$. Then the (s,t)-entry of $BY(i,j)B^{-1}$ is equal to yb_{ss}/b_{tt}, where $B = (b_{**})$ and y is the (s,t)-entry of $Y(i,j)$. In other words, y is semi-invariant. Remark that $(s,t) > (i,j)$. The (s,t)-entry y' of $X(s,t)$ is a unit, and $a_{st}(i,j) := y/y'$ is B_n-invariant. The new matrix $Y' := Y(i,j) - a_{st}(i,j)X(s,t)$ is a canonical matrix and $\mathrm{Supp}Y' \subseteq \overline{(i,j)} \cup (J \setminus \{(s,t)\})$. Instead of $Y(i,j)$ and J, we consider Y' and $J' := \overline{\mathrm{Supp}Y'} \setminus \overline{(i,j)}$. Then $J' = \emptyset$ or the minimal element of J' is bigger than (s,t). By induction on the minimum elements of J', we can obtain B_n-invariants $a_{k,\ell}(i,j)$ for $(k,\ell) \in J$ such that $X(i,j) := Y(i,j) - \sum_{(k,\ell)} a_{k\ell}(i,j)X(k,\ell)$ is an (i,j)-shaped canonical matrix. By repeating this discussion, we can obtain an (i,j)-shaped canonical matrix $X(i,j)$ for each $(i,j) \in \mathcal{I}_n$. $\square$

From the lemma above, we obtained well-shaped canonical matrices $X(i,j)$ from $Y(i,j) = \eta_{i'j'}(\alpha_{ij})$. For proving Lemma 2.5, we only need to verify that there exists $Q \in \tilde{B}_n(R)$ such that $QX(i,j)Q^{-1} \in \mathcal{B}_n(R^{\mathrm{ch}})$. The next lemma is useful for the discussion below.

Lemma 2.10. *Let X be a canonical matrix of $\mathcal{B}_n(R)$. Set $J := \overline{\mathrm{Supp} X}$. Then there exist* B_n*-invariants a_{ij} for $(i,j) \in J$ such that*

$$X = \sum_{(i,j)\in J} a_{ij} X(i,j).$$

Proof. If $J = \emptyset$, then the statement is trivial. Suppose that $J \neq \emptyset$. Let $(i,j) \in J$ be the minimum element of J. The (i,j)-entry x_{ij} of X is semi-invariant, and hence $a_{ij} := x_{ij}/X(i,j)_{ij}$ is B_n-invariant. Here we denote by $X(i,j)_{ij}$ the (i,j)-entry of $X(i,j)$. The new matrix $X' := X - a_{ij}X_{ij}$ is a canonical matrix. If $X' \neq 0$, then $J' := \overline{\mathrm{Supp} X'} \subset J$ and the minimum element of J' is bigger than (i,j). By induction on the minimum of $J = \overline{\mathrm{Supp} X}$, we can prove the statement. $\square$

Definition 2.11. Let R' be the subalgebra of R over R^{ch} generated by the following elements:

$$C := \left\{ \begin{array}{cl} (X(1,i)_{1i})^{\pm 1} & ([\frac{n+1}{2}] + 1 \leq i \leq n), \\ (X(i,n)_{in})^{\pm 1} & (2 \leq i \leq [\frac{n+1}{2}]), \\ X(1,1)_{1i} & (2 \leq i \leq n), \\ X(i,i)_{ji} & (3 \leq i \leq n,\ 2 \leq j \leq i-1) \end{array} \right\}.$$

Lemma 2.12. *For each $(i,j) \in \mathcal{I}_n$, $X(i,j) \in \mathcal{B}_n(R')$.*

Proof. Note that $X(i,i)_{ii} \in (R^{\mathrm{ch}})^\times$ for $1 \leq i \leq n$. It is easy to see that $X(1,1), X(1,n) \in \mathcal{B}_n(R')$. First, we verify that $X(n,n) \in \mathcal{B}_n(R')$. Remark that $X(n,n)_{in} \in C$ for $2 \leq i \leq n-1$ and $X(n,n)_{nn} \in (R^{\mathrm{ch}})^\times$. For proving that $X(n,n)_{1n} \in R'$, let us consider the canonical matrix $X(1,1)X(n,n)$. By Lemma 2.10, we see that $X(1,1)X(n,n) = a(1,n)X(1,n)$ for some $a(1,n) \in R^{\mathrm{ch}}$ because $\mathrm{Supp}(X(1,1)X(n,n)) \subseteq \{(1,n)\}$. Comparing the $(1,n)$-entries, we have

$$X(1,1)_{11}X(n,n)_{1n} + \sum_{k=2}^{n} X(1,1)_{1k}X(n,n)_{kn} = a(1,n)X(1,n)_{1n}.$$

Hence we see that

$$X(n,n)_{1n} = X(1,1)_{11}^{-1}(a(1,n)X(1,n)_{1n} - \sum_{k=2}^{n} X(1,1)_{1k}X(n,n)_{kn})$$

and that $X(n,n)_{1n} \in R'$. Therefore we have $X(n,n) \in \mathcal{B}_n(R')$.

Next, we show that $X(1,i)_{1i} \in (R')^\times$ for $2 \leq i \leq n$ and that $X(i,n)_{in} \in (R')^\times$ for $2 \leq i \leq n$. Let us consider the canonical matrix $X(1,i)X(i,n)$. The support is contained in $\{(1,n)\}$, and $X(1,i)X(i,n) = aX(1,n)$ for some $a \in R^{\mathrm{ch}}$. Comparing the $(1,n)$-entries, we have

$$X(1,i)_{1i}X(i,n)_{in} = aX(1,n)_{1n}.$$

Since $X(1,i)_{1i}, X(i,n)_{in}, X(1,n)_{1n} \in R^{\times}$, we have $a \in (R^{\rm ch})^{\times}$. Because $(X(1,i)_{1i})^{\pm 1} \in C$ for $[\frac{n+1}{2}]+1 \leq i \leq n$ and $(X(i,n)_{in})^{\pm 1} \in C$ for $2 \leq i \leq [\frac{n+1}{2}]$, we see that $X(1,i)_{1i} \in (R')^{\times}$ for $2 \leq i \leq n$ and that $X(i,n)_{in} \in (R')^{\times}$ for $2 \leq i \leq n$.

Third, we show that $X(1,i) \in \mathcal{B}_n(R')$ for $1 < i < n$. We have known that $X(1,n) \in \mathcal{B}_n(R')$. Let us consider the canonical matrices $X(1,i)X(j,j)$ for $j > i$. By Lemma 2.10 there exist B_n-invariants $a(i,k)$ for $j \leq k \leq n$ such that $X(1,i)X(j,j) = a(1,j)X(1,j) + \sum_{k>j} a(1,k)X(1,k)$ since $\mathrm{Supp}X(1,i)X(j,j) \subseteq \overline{\{(1,j)\}}$. Comparing the $(1,j)$-entries, we have

$$\begin{aligned} X(1,i)_{1i}X(j,j)_{ij} + X(1,i)_{1,i+1}X(j,j)_{i+1,j} + \cdots + X(1,i)_{1j}X(j,j)_{jj} \\ = a(1,j)X(1,j)_{1j}. \end{aligned}$$

We have seen that $X(1,i)_{1i} \in R'$. Assume that $X(1,i)_{1k} \in R'$ for $i \leq k \leq j-1$. By the equality above, we have

$$\begin{aligned} X(1,i)_{1j} = \ & X(j,j)_{jj}^{-1}(a(1,j)X(1,j)_{1j} - X(1,i)_{1i}X(j,j)_{ij} - \cdots \\ & -X(1,i)_{1,j-1}X(j,j)_{j-1,j}). \end{aligned}$$

Since $X(j,j)_{ij}, \ldots, X(j,j)_{j-1,j} \in C$ and $X(1,j)_{1j} \in R'$, we obtain $X(1,i)_{1j} \in R'$. By induction on j, we have $X(1,i)_{1j} \in R'$ for $i \leq j \leq n$. Thus $X(1,i) \in \mathcal{B}_n(R')$.

Finally, we show that $X(i,j) \in \mathcal{B}_n(R')$ for each $(i,j) \in \mathcal{I}_n$. If $i=1$, then we have checked it. Hence we may assume that $i > 1$. By Lemma 2.10 we see that

$$X(1,i)X(i,j) = a(1,j)X(1,j) + \cdots + a(1,n)X(1,n)$$

for suitable B_n-invariants $a(1,k)$ $(j \leq k \leq n)$. In particular, we obtain $X(1,i)X(i,j) \in \mathcal{B}_n(R')$. The $(1,k)$-entry of $X(1,i)X(i,j)$ is $X(1,i)_{1i}X(i,j)_{ik}$ for $j \leq k \leq n$. Since $X(1,i)_{1i} \in (R')^{\times}$, $X(i,j)_{ik} \in R'$. Suppose that there exists $1 \leq \ell \leq i-1$ such that $X(i,j)_{mk} \in R'$ for $m > \ell$ and $j \leq k \leq n$. Then we prove that $X(i,j)_{\ell k} \in R'$ for $j \leq k \leq n$. By using Lemma 2.10 again, we see that

$$X(1,\ell)X(i,j) = a(1,j)X(1,j) + \cdots + a(1,n)X(1,n) \in \mathcal{B}_n(R')$$

for suitable B_n-invariants $a(1,k)$ $(j \leq k \leq n)$. For $j \leq k \leq n$, the $(1,k)$-entry of $X(1,\ell)X(i,j)$ is

$$X(1,\ell)_{1\ell}X(i,j)_{\ell k} + X(1,\ell)_{1,\ell+1}X(i,j)_{\ell+1,k} + \cdots + X(1,\ell)_{1i}X(i,j)_{ik} \in R'.$$

Since $X(1,\ell)_{1\ell} \in (R')^{\times}$ and $X(1,\ell) \in \mathcal{B}_n(R')$, we have $X(i,j)_{\ell k} \in R'$ for $j \leq k \leq n$ by the hypothesis. By induction on ℓ, we see that $X(i,j)_{\ell k} \in R'$ for $1 \leq \ell \leq i$ and $j \leq k \leq n$. Therefore we proved that $X(i,j) \in \mathcal{B}_n(R')$ for each $(i,j) \in \mathcal{I}_n$. $\square$

Now we can prove Lemma 2.5.

Proof of Lemma 2.5. By the long discussion above, we only need to show that there exists $Q \in \tilde{B}_n(R)$ such that $QX(i,j)Q^{-1} \in \mathcal{B}_n(R^{\mathrm{ch}})$. For $\{X(i,j) \mid (i,j) \in \mathcal{I}_n\}$ we introduced the set C in Definition 2.11. Let $Q \in \tilde{B}_n(R)$. For $\{QX(i,j)Q^{-1} \mid (i,j) \in \mathcal{I}_n\}$ we define the set Q^*C in a similar way. From now on, we prove that there exists $Q \in \tilde{B}_n(R)$ such that any element of Q^*C is 0 or 1. More precisely, we prove that

$$\begin{aligned}
(QX(1,i)Q^{-1})_{1i} &= 1 \quad ([\frac{n+1}{2}]+1 \leq i \leq n) \\
(QX(i,n)Q^{-1})_{in} &= 1 \quad (2 \leq i \leq [\frac{n+1}{2}]) \\
(QX(1,1)Q^{-1})_{1i} &= 0 \quad (2 \leq i \leq n) \\
(QX(i,i)Q^{-1})_{ji} &= 0 \quad (3 \leq i \leq n, 2 \leq j \leq i-1).
\end{aligned}$$

Let us find $Q = (q_{ij}) \in \tilde{B}_n(R)$. First, we set

$$(q_{11}, q_{22}, \ldots, q_{nn}) := (1, \frac{\tau_{1,n}}{\tau_{2,n}}, \frac{\tau_{1,n}}{\tau_{3,n}}, \ldots, \frac{\tau_{1,n}}{\tau_{[\frac{n+1}{2}],n}}, \tau_{1,[\frac{n+1}{2}]+1}, \ldots, \tau_{1,n-1}, \tau_{1,n}).$$

Here recall that $\tau_{ij} = \eta_{i'j'}(\alpha_{ij})_{ij} = Y(i,j)_{ij} = X(i,j)_{ij}$. Then it is easy to see that $(QX(1,i)Q^{-1})_{1i} = 1$ for $[\frac{n+1}{2}] + 1 \leq i \leq n$ and $(QX(i,n)Q^{-1})_{in} = 1$ for $2 \leq i \leq [\frac{n+1}{2}]$.

Next, let us define $q_{ij} \in R$ for $i < j$. Set $Q^{-1} = (q'_{**})$. Let $X = (x_{**}) \in \mathcal{B}_n(R)$. From $QQ^{-1} = I_n$, we have $q'_{kk} = q_{kk}^{-1}$ and

$$q_{ij}q'_{jj} + q_{i,j-1}q'_{j-1,j} + \cdots + q_{i,i+1}q'_{i+1,j} + q_{ii}q'_{ij} = 0$$

for $i < j$. Since

$$q'_{ij} = -q_{ii}^{-1}(q_{ij}q'_{jj} + q_{i,j-1}q'_{j-1,j} + \cdots + q_{i,i+1}q'_{i+1,j}),$$

we see that q'_{ij} can be expressed by $\{q_{k\ell} \mid (k,\ell) \leq (i,j)\}$. The (i,j)-entry of QXQ^{-1} is

$$\begin{aligned}
& q_{ij}x_{jj}q'_{jj} + q_{ii}x_{ii}q'_{ij} + \sum_{(k,\ell)\neq(i,i),(j,j)} q_{ik}x_{k\ell}q'_{\ell j} \\
= \;& (x_{jj} - x_{ii})q_{jj}^{-1}q_{ij} + (\text{ rational function of } \{q_{k\ell} \mid (k,\ell) < (i,j)\}).
\end{aligned}$$

Assume that $q_{k\ell} \in R$ is defined for each $(k,\ell) < (i,j)$. Now we define $q_{ij} \in R$. If $i = 1$, then put $X = X(1,1)$. Since $x_{11} \in R^*$ and $x_{jj} = 0$, the $(1,j)$-entry of $QX(1,1)Q^{-1}$ is $-x_{11}q_{jj}^{-1}q_{ij} +$(lower term), and hence we can find q_{ij} satisfying the equation $(QX(1,1)Q^{-1})_{1j} = 0$. If $i > 1$, then put $X = X(j,j)$. Since $x_{ii} = 0$ and $x_{jj} \in R^*$, the (i,j)-entry of $QX(j,j)Q^{-1}$ is $x_{jj}q_{jj}^{-1}q_{ij}$ + (lower term), and hence we can find q_{ij}

satisfying the equation $(QX(j,j)Q^{-1})_{ij} = 0$. By induction on $(i,j) \in \mathcal{I}_n$, we can define $Q = (q_{ij})$ satisfying the equations. In particular, any element of Q^*C is 0 or 1.

By Lemma 2.12 we have $X(i,j) \in \mathcal{B}_n(R')$. Similarly, we see that all entries of $QX(i,j)Q^{-1}$ are contained in the algebra generated by Q^*C over R^{ch}. Since any element of Q^*C is 0 or 1, $QX(i,j)Q^{-1} \in \mathcal{B}_n(R^{\mathrm{ch}})$ for $(i,j) \in \mathcal{I}_n$. $\square$

Now we can finish the proof of Lemma 2.2.

Proof of Lemma 2.2. By Lemma 2.5 there exists $Q \in \tilde{B}_n(R)$ such that $Q\rho(\gamma)Q^{-1} \in \mathcal{B}_n(R^{\mathrm{ch}})$ for each $\gamma \in \Gamma$. The inclusion $R^{\mathrm{ch}} \to R$ induces the morphism $\pi : \mathrm{B}_n(\Gamma)_{B,A} \to \mathrm{Ch}_n(\Gamma)_{B,A}$. We have the section $s : \mathrm{Ch}_n(\Gamma)_{B,A} \to \mathrm{B}_n(\Gamma)_{B,A}$ associated to the $\mathcal{B}_n$-representation $\rho' := Q\rho Q^{-1} : \Gamma \to \mathcal{B}_n(R^{\mathrm{ch}})$ with Borel mold.

We show that the morphism $\phi : \mathrm{Ch}_n(\Gamma)_{B,A} \times \mathrm{B}_n \to \mathrm{B}_n(\Gamma)_{B,A}$ associated to the $\mathcal{B}_n$-representation $\tilde{Q}\rho'\tilde{Q}^{-1}$ gives an isomorphism. Here we denote by $\tilde{Q}$ the universal matrix of B_n. For a scheme Z, the morphism $\phi_*(Z) : h_{\mathrm{Ch}_n(\Gamma)_{B,A}}(Z) \times h_{\mathrm{B}_n}(Z) \to h_{\mathrm{B}_n(\Gamma)_{B,A}}(Z)$ is injective because of Proposition 1.11. Let us prove that $\phi_*(Z)$ is surjective. Let $\psi \in h_{\mathrm{B}_n(\Gamma)_{B,A}}(Z)$. Put $\chi := \pi_*(Z)(\psi) \in h_{\mathrm{Ch}_n(\Gamma)_{B,A}}(Z)$ and $\psi' := s_*(Z)(\chi) \in h_{\mathrm{B}_n(\Gamma)_{B,A}}(Z)$. Note that $\pi_*(Z)(\psi) = \pi_*(Z)(\psi') = \chi$. For ψ and ψ', we can define (i,j)-shaped canonical matrices $X(i,j)$ and $X'(i,j)$ on Z, respectively. In a similar way as in the proof of Lemma 2.5, we see that there exists $Q \in h_{\mathrm{B}_n}(Z)$ such that $C = Q^*C'$, where we define C and C' as in Definition 2.11 for ψ and ψ', respectively. Since the B_n-invariants are same for ψ and ψ', $X(i,j) = QX'(i,j)Q^{-1}$ and $\psi = Q\psi'Q^{-1}$. In particular, $\phi_*(Z)(\psi', Q) = \psi$ and hence $\phi_*(Z)$ is surjective. Therefore ϕ is an isomorphism.

The morphism $\mathrm{B}_n(\Gamma)_{B,A} \to \mathrm{Ch}_n(\Gamma)_{B,A}$ gives a B_n-principal fiber bundle. We can check that the functor $\mathcal{E}qBB_n(\Gamma)_A$ is representable by $\mathrm{Ch}_n(\Gamma)_{B,A}$. This completes the proof of Lemma 2.2. $\square$

By Corollary 2.3, we see that $\mathcal{E}qBB_n(\Gamma)$ is representable. We introduce the following definition.

Definition 2.13. The scheme $\mathrm{Ch}_n(\Gamma)_B$ which represents the functor $\mathcal{E}qBB_n(\Gamma) = \mathcal{E}qB_n(\Gamma)$ is called the *moduli of representations with Borel mold* of degree n for Γ. It is also called the *character variety with Borel mold* of degree n for Γ.

Remark 2.14. The canonical morphism $\pi : \mathrm{Rep}_n(\Gamma)_B \to \mathrm{Ch}_n(\Gamma)_B$ is a principal fiber bundle with fiber PGL_n. The canonical morphism

$\pi' : \mathrm{B}_n(\Gamma)_B \to \mathrm{Ch}_n(\Gamma)_B$ is also a principal fiber bundle with fiber B_n. These principal fiber bundles have a local trivialization with respect to Zariski topology. They are universal geometric quotient in [3].

The construction of the moduli of representations with Borel mold gives us the following diagram:

$$\begin{array}{ccc} \mathrm{B}_n(\Gamma)_B \times \mathrm{PGL}_n & \xrightarrow{f} & \mathrm{Rep}_n(\Gamma)_B \\ \downarrow p_1 & & \downarrow \pi \\ \mathrm{B}_n(\Gamma)_B & \xrightarrow{\pi'} & \mathrm{Ch}_n(\Gamma)_B, \end{array}$$

where

$$\begin{array}{rccc} f & : & \mathrm{B}_n(\Gamma)_B \times \mathrm{PGL}_n & \to & \mathrm{Rep}_n(\Gamma)_B \\ & & (\rho, Q) & \mapsto & Q^{-1}\rho Q \end{array}$$

and $p_1 : \mathrm{B}_n(\Gamma)_B \times \mathrm{PGL}_n \to \mathrm{B}_n(\Gamma)_B$ is the first projection. The morphism f is a principal fiber bundle with fiber B_n which has a local trivialization with respect to Zariski topology.

Lemma 2.15. *Let Γ be a finitely generated group or monoid. Then $\mathrm{Ch}_n(\Gamma)_B$ is of finite type over $\mathbb{Z}$.*

Proof. Since the representation variety $\mathrm{Rep}_n(\Gamma)$ is of finite type over $\mathbb{Z}$ when Γ is finitely generated, so is a subscheme $\mathrm{Rep}_n(\Gamma)_B$. The principal fiber bundle $\pi : \mathrm{Rep}_n(\Gamma)_B \to \mathrm{Ch}_n(\Gamma)_B$ with fiber PGL_n has a local trivialization with respect to Zariski topology, and hence $\mathrm{Ch}_n(\Gamma)_B$ is of finite type over $\mathbb{Z}$. □

Remark 2.16. Lemma 2.15 can be verified by investigating the invariants directly. Since $\mathrm{Rep}_n(\Gamma)_B$ is quasi-compact, $\mathrm{Ch}_n(\Gamma)_B$ is also quasi-compact. It is essential to prove that $\mathrm{Ch}_n(\Gamma)_B$ is locally of finite type over $\mathbb{Z}$. Let R^{ch} be the affine ring of $\mathrm{Ch}_n(\Gamma)_{B,A}$. Let $\rho' : \Gamma \to \mathcal{B}_n(R^{\mathrm{ch}})$ be the $\mathcal{B}_n$-representation with Borel mold in the proof of Lemma 2.2. For a generator $\{\alpha_i\}_{i=1}^N$ of Γ, we consider the set S of all entries of $\rho'(\alpha_i)$ for $i = 1, 2, \ldots, N$. Then R^{ch} is generated by S over $\mathbb{Z}$. Indeed, let R_0 be the subalgebra of R^{ch} generated by S over $\mathbb{Z}$. We can define the $\mathcal{B}_n$-representation $\rho'' : \Gamma \to \mathcal{B}_n(R_0)$ with Borel mold such that $\rho'' \otimes_{R_0} R^{\mathrm{ch}} = \rho'$. We define the section of $\mathrm{B}_n(\Gamma)_{B,A} \to \mathrm{Ch}_n(\Gamma)_{B,A} \to \mathrm{Spec} R_0$ associated to ρ''. We can show that for $f \in \mathrm{Hom}(\mathrm{Ch}_n(\Gamma)_{B,A}, \mathbb{A}^1_{\mathbb{Z}})$ there exists a unique $f' \in \mathrm{Hom}(\mathrm{Spec} R_0, \mathbb{A}^1_{\mathbb{Z}})$ such that $\mathrm{Ch}_n(\Gamma)_{B,A} \to \mathrm{Spec} R_0 \xrightarrow{f'} \mathbb{A}^1_{\mathbb{Z}}$ is f. Hence we have $R^{\mathrm{ch}} = R_0$. Therefore R^{ch} is finitely generated over $\mathbb{Z}$.

Proposition 2.17. *The moduli* $\mathrm{Ch}_n(\Gamma)_B$ *is a separated scheme over* $\mathbb{Z}$*.*

Before proving Proposition 2.17, we introduce the following lemma. We define the subgroup scheme $\tilde{B}_n$ of GL_n by $\tilde{B}_n := \{(b_{ij}) \in \mathrm{GL}_n \mid b_{ij} = 0 \text{ for each } i > j\}$.

Lemma 2.18. *Let* R *be a valuation ring and* K *its quotient field. Suppose that* $P \in \tilde{B}_n(K)$ *satisfies* $P\mathcal{B}_n(R)P^{-1} = \mathcal{B}_n(R)$*. Then there exist* $\lambda \in K$ *and* $Q \in \tilde{B}_n(R)$ *such that* $P = \lambda Q$*.*

Proof. Set $P = (p_{ij})$. Let v be a valuation of R. We claim that $v(p_{11}) = v(p_{22}) = \cdots = v(p_{nn})$ and that $v(p_{ij}) \geq v(p_{11})$. From this claim, $\lambda := p_{11}$ and $Q := 1/p_{11} \cdot P$ are what we want, and hence the statement can be proved.

By the assumption, $PE_{ij}P^{-1} \in \mathcal{B}_n(R)$, and hence the (i,j)-entry $p_{ii}/p_{jj} \in R$. Since $P^{-1}\mathcal{B}_n(R)P = \mathcal{B}_n(R)$ also holds, the (i,j)-entry p_{jj}/p_{ii} of $P^{-1}E_{ij}P$ is contained in R. Therefore $v(p_{ii}) = v(p_{jj})$. For each $i < j$, the (i,j)-entry p_{ij}/p_{jj} of $PE_{jj}P^{-1}$ is contained in R, which conclude that $v(p_{ij}) \geq v(p_{jj}) = v(p_{11})$. Thereby we have proved the claim. □

Proof of Proposition 2.17. We prove that $\mathrm{Ch}_n(\Gamma)_B$ is separated by using valuative criterion. Let R be a valuation ring and K be its quotient field. Suppose that $[\rho]$ and $[\rho']$ be two R-valued points of $\mathrm{Ch}_n(\Gamma)_B$ which coincide as K-valued points. We show that $[\rho] = [\rho']$ as R-valued points. Let us take representatives $\rho, \rho' : \Gamma \to \mathcal{B}_n(R)$ of $[\rho], [\rho']$, respectively. Since ρ, ρ' are equivalent to each other over K, there exists $P \in \tilde{B}_n(K)$ such that $P\rho P^{-1} = \rho'$. The algebra $\mathcal{B}_n(R)$ is generated by the image of ρ or ρ' over R, and hence $P\mathcal{B}_n(R)P^{-1} = \mathcal{B}_n(R)$. By Lemma 2.18, there exist $\lambda \in K$ and $Q \in \tilde{B}_n(R)$ such that $P = \lambda Q$. We obtain $Q\rho Q^{-1} = \rho'$, and we conclude that ρ and ρ' are equivalent over R. □

Summarizing the above discussion, we obtain:

Theorem 2.19. *Let* Γ *be a group or a monoid. The sheafification with respect to Zariski topology of the following functor is representable by a separated scheme* $\mathrm{Ch}_n(\Gamma)_B$ *over* $\mathbb{Z}$*:*

$$\begin{array}{rcccl} \mathcal{E}q{B_n}(\Gamma) & : & (\mathbf{Sch}) & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{ \text{ rep. with Borel mold of deg } n \text{ for } \Gamma \ \}/\sim \end{array}$$

If Γ *is finitely generated, then* $\mathrm{Ch}_n(\Gamma)_B$ *is of finite type over* $\mathbb{Z}$*.*

Remark 2.20. In this paper, we deal with only representations of groups or monoids. However, we can construct the moduli of representations with Borel mold for an arbitrary associative algebra. Let A be an associative algebra over a commutative ring R. We define a reprensentation with Borel mold for A on a scheme over R in a similar way as the group case. Then we can construct the moduli scheme of representations with Borel mold separated over R. If A is a finitely generated algebra over R, then the moduli is of finite type over R. (The fact that the moduli is quasi-compact follows from that there exist a noetherian subring S of R and a finitely generated subalgebra A_0 of A over S such that $A_0 \otimes_S R \to A$ is surjective and the morphism $\mathrm{B}_n(A)_B \to \mathrm{B}_n(A_0)_B$ is affine and hence quasi-compact.)

3. Basic results

In this section we introduce basic results on the moduli of representations with Borel mold.

Let Γ be a group or a monoid. Let ρ be a representation with Borel mold for Γ on a scheme X. Let us define the action of Γ on the trivial vector bundle $\mathcal{O}_X^{\oplus n}$ by $\Gamma \xrightarrow{\rho} \mathrm{M}_n(\Gamma(X, \mathcal{O}_X)) = \mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n})$. Then we have the following proposition.

Proposition 3.1. *For each $1 < i < n$, there exists a unique Γ-invariant subbundle $\mathcal{E}_i \subseteq \mathcal{O}_X^{\oplus n}$ of rank i on X. The Γ-invariant subbundles $0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_{n-1} \subseteq \mathcal{O}_X^{\oplus n}$ form a complete flag of $\mathcal{O}_X^{\oplus n}$.*

Proof. Lemma 1.9 follows the uniqueness of Γ-invariant subbundles. Since we have Γ-invariant subbundles locally, by gluing them together we obtain unique Γ-invariant subbundles globally. □

From the above proposition, we easily obtain:

Theorem 3.2. *The representation variety with Borel mold $\mathrm{Rep}_n(\Gamma)_B$ has a unique complete flag of Γ-invariant subbundles $0 \subseteq \mathcal{E}_{\Gamma,n}^{(1)} \subseteq \cdots \subseteq \mathcal{E}_{\Gamma,n}^{(n-1)} \subseteq \mathcal{O}_{\mathrm{Rep}_n(\Gamma)_B}^{\oplus n}$ which has the universal property: for any scheme X and for any representation ρ of degree n with Borel mold for Γ on X, the unique Γ-invariant subbundles $\mathcal{E}_i$ of rank i on X is obtained as $f^*\mathcal{E}_{\Gamma,n}^{(i)}$, where $f : X \to \mathrm{Rep}_n(\Gamma)_B$ is the morphism associated to ρ.*

Remark 3.3. For a representation with Borel mold ρ for Γ on a scheme X, we have unique Γ-invariant subbundles $0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_{n-1} \subseteq \mathcal{O}_X^{\oplus n}$ on X by Proposition 3.1. The action of Γ on $\mathcal{E}_i/\mathcal{E}_{i-1}$ induces the character χ_i of Γ for each $1 \leq i \leq n$. The correspondence $\rho \mapsto (\chi_1, \chi_2, \dots, \chi_n)$ gives us a morphism $\mathrm{Ch}_n(\Gamma)_B \to \mathrm{Ch}_1(\Gamma) \times \cdots \times \mathrm{Ch}_1(\Gamma)$. Here $\mathrm{Ch}_1(\Gamma)$ is the moduli of characters for Γ.

In the case $n = 2$, we have a morphism $\mathrm{Ch}_2(\Gamma)_B \to \mathrm{Ch}_1(\Gamma) \times \mathrm{Ch}_1(\Gamma)$. The fiber at (χ_1, χ_2) is the projective space of the extension classes of characters (χ_1, χ_2). However, in this article we will not go into details on the relation between the moduli of representations with Borel mold and the extension classes of characters.

We denote by $\mathrm{Rep}_n(m)_B$ the representation variety with Borel mold for the free monoid of rank m. The following proposition follows that $\mathrm{Rep}_n(m)_B$ contains the representation variety with Borel mold $\mathrm{Rep}_n(\mathrm{F}_m)_B$ for the free group of rank m as an open subscheme.

Proposition 3.4. *Let $\Upsilon_m = \langle \alpha_1, \dots, \alpha_m \rangle$ be the free monoid of rank m. Let $\mathrm{F}_m = \langle \alpha_1, \dots, \alpha_m \rangle$ be the free group of rank m. The inclusion $\Upsilon_m \to \mathrm{F}_m$ by $\alpha_i \to \alpha_i$ induces an open immersion $\mathrm{Rep}_n(\mathrm{F}_m)_B \to \mathrm{Rep}_n(m)_B$.*

Proof. Restricting each representation ρ with Borel mold for F_m to the free monoid Υ_m, we can obtain a morphism $\mathrm{Rep}_n(\mathrm{F}_m)_B \to \mathrm{Rep}_n(m)_B$. Indeed, by the Cayley-Hamilton theorem, $\rho(\alpha_i^{-1})$ is expressed as a polynomial of $\rho(\alpha_i)$, and hence $\langle \rho(\alpha_1), \dots, \rho(\alpha_m) \rangle$ generates a Borel mold. It is easy to check that the morphism $\mathrm{Rep}_n(\mathrm{F}_m)_B \to \mathrm{Rep}_n(m)_B$ is an open immersion. $\square$

In the case $n = 1$, we see that $\mathrm{Rep}_1(m) \cong \mathbb{A}_{\mathbb{Z}}^m$ and $\mathrm{Rep}_1(F_m) \cong (\mathbb{A}_{\mathbb{Z}} \setminus \{0\})^m$. In the case $n \geq 2$ and $m = 1$, we also see that $\mathrm{Rep}_n(1) = \emptyset$ and $\mathrm{Rep}_n(F_1) = \emptyset$. In the case $n \geq 2$ and $m \geq 2$, $\mathrm{Rep}_n(m)_B$ and $\mathrm{Rep}_n(\mathrm{F}_m)_B$ are non-empty (see [6]). Furthermore we have:

Proposition 3.5. *Let $n \geq 2$ and $m \geq 2$. The scheme $\mathrm{Rep}_n(m)_B$ is smooth over $\mathbb{Z}$. In particular, $\mathrm{Rep}_n(\mathrm{F}_m)_B$ is smooth over $\mathbb{Z}$.*

Proof. Let A be an artinian local ring and let I be an ideal of A with $I^2 = 0$. Let $\overline{\rho} \in \mathrm{Rep}_n(m)_B(A/I)$. Then we show that there exists $\rho \in \mathrm{Rep}_n(m)_B(A)$ such that the reduction of ρ to A/I is equal to $\overline{\rho}$. We take a system of free generators $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ of the free monoid Υ_m. There exists $\overline{P} \in \mathrm{GL}_n(A/I)$ such that $\overline{P}^{-1}\overline{\rho}(\alpha_i)\overline{P}$ is an upper triangular matrix for each $1 \leq i \leq m$. Take $P \in \mathrm{GL}_n(A)$ such that the reduction to A/I is equal to $\overline{P}$. We also take an upper

triangular matrix $X_i \in \mathrm{GL}_n(A)$ as a lift of $\overline{P}^{-1}\overline{\rho}(\alpha_i)\overline{P}$ for each i. Then we define the representation $\rho : \Upsilon_m \to \mathrm{GL}_n(A)$ by $\rho(\alpha_i) := PX_iP^{-1}$ for each $1 \leq i \leq m$. We easily see that ρ is the desired representation with Borel mold, and hence we have proved the statement. □

Corollary 3.6. *For $m \geq 2$, the moduli scheme of representations with Borel mold $\mathrm{Ch}_n(m)_B$ for the free monoid Υ_m is smooth over $\mathbb{Z}$. In particular, the open subscheme $\mathrm{Ch}_n(\mathrm{F}_m)_B$ of $\mathrm{Ch}_n(m)_B$ is also smooth over $\mathbb{Z}$.*

Proof. The quotient morphism $\mathrm{Rep}_n(m)_B \to \mathrm{Ch}_n(m)_B$ gives a PGL_n-principal fiber bundle. Since $\mathrm{Rep}_n(m)_B$ is smooth over $\mathbb{Z}$, so is $\mathrm{Ch}_n(m)_B$. □

Remark 3.7. In [6] we have proved that $\mathrm{Ch}_n(m)_B$ is a fibre bundle over the configuration space $F_n(\mathbb{A}^m_{\mathbb{Z}})$ of the affine space $\mathbb{A}^m_{\mathbb{Z}}$ with fibre $(\mathbb{P}^{m-2}_{\mathbb{Z}})^{n-1} \times (\mathbb{A}^{m-1}_{\mathbb{Z}})^{(n-2)(n-1)/2}$ with respect to Zariski topology. In particular, the rational function field of $\mathrm{Ch}_n(m)_B$ is rational over $\mathbb{Q}$ if $n = 1$ or $n, m \geq 2$. Furthermore, if k is a field, then $\mathrm{Ch}_n(m)_B \otimes k$ is a smooth rational variety over k.

4. The degree 2 case

In this section, we deal with representations of degree 2 with Borel mold. In the degree 2 case, a mold is a Borel mold if and only if it has rank 3.

The following proposition gives us one of characterizations of representations of degree 2 with Borel mold for a group Γ.

Proposition 4.1. *Let Γ be a group. Let ρ be a representation of degree 2 for Γ on a scheme X. Then ρ is a representation with Borel mold if and only if the following two conditions hold:*

(i) $\mathrm{tr}(\rho([\alpha, \beta])) = 2$ *for each $\alpha, \beta \in \Gamma$, where $[\alpha, \beta] := \alpha\beta\alpha^{-1}\beta^{-1}$.*

(ii) *the image of the composition $\Gamma \xrightarrow{\rho} \mathrm{GL}_2(\Gamma(X, \mathcal{O}_X)) \to \mathrm{GL}_2(k(x))$ is not an abelian group for each point $x \in X$, where $k(x)$ is the residue field of x.*

Proof. If ρ is a representation with Borel mold, then for each $x \in X$ there exist a neighborhood U of x and $P \in \mathrm{GL}_2(\mathcal{O}_X(U))$ such that $P^{-1}(\rho\mid_U)(\gamma)P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ for each $\gamma \in \Gamma$. Hence we have $\mathrm{tr}(\rho([\alpha, \beta])) = 2$ for each $\alpha, \beta \in \Gamma$. Since for the mold $\mathcal{O}_X[\rho(\Gamma)] \otimes k(x)$ is a non-commutative algebra for $x \in X$, the condition (ii) holds.

Conversely suppose that two conditions (i) and (ii) hold. Then we show that the subsheaf $\mathcal{O}_X[\rho(\Gamma)]$ of $\mathrm{M}_2(\mathcal{O}_X)$ is a rank 3 mold. If $\mathcal{O}_X[\rho(\Gamma)]$ is a rank 3 mold, then it is a Borel mold from Corollary 1.17, which completes the proof. For each $x \in X$, there exists $\alpha, \beta \in \Gamma$ such that $\rho(\alpha)$ and $\rho(\beta)$ are not commutative as elements of $\mathrm{GL}_2(k(x))$. From the assumption, the discriminant $\Delta(\rho(\alpha), \rho(\beta))$ in Definition 5.1 is equal to 0, since $\Delta(\rho(\alpha), \rho(\beta)) = \det(\rho(\alpha\beta))(\mathrm{tr}(\rho([\alpha, \beta]) - 2)$ by Proposition 5.3. Proposition 5.4 follows that the subsheaf of $\mathcal{O}_U$-algebras $\mathcal{O}_U[(\rho\,|_U)(\alpha), (\rho\,|_U)(\beta)]$ generated by $(\rho\,|_U)(\alpha), (\rho\,|_U)(\beta)$ is a rank 3 mold on some affine neighbourhood U of x. For each $\gamma \in \Gamma$, we only have to show that $(\rho|_U)(\gamma) \in \mathcal{O}_U[(\rho|_U)(\alpha), (\rho|_U)(\beta)]$. By considering the subgroup of Γ generated by α, β, and γ, we have reduced to the case that Γ is a finitely generated group. Let $U = \mathrm{Spec}(R)$. Let us denote $\rho\,|_U$ by ρ. By changing R to the subring of R generated by all the entries of $(\rho|_U)(\alpha), (\rho|_U)(\beta)$, and $(\rho|_U)(\gamma)$ over $\mathbb{Z}$ (if necessary, we may add more finitely many elements to the subring), we may assume that R is a noetherian ring. Since we only need to prove that $(\rho|_U)(\gamma) \in R[(\rho|_U)(\alpha), (\rho|_U)(\beta)]$ on a neighbourhood of each point of $\mathrm{Spec}R$, we may also assume that R is a noetherian local ring.

First suppose that R is a reduced noetherian local ring. Since the subalgebra $R[\rho(\alpha), \rho(\beta)]$ of $\mathrm{M}_2(R)$ is a Borel mold, there exists $P \in \mathrm{GL}_2(R)$ such that $P^{-1}\rho(\alpha)P$ and $P^{-1}\rho(\beta)P$ generate the algebra $\mathcal{B}_2 \otimes_{\mathbb{Z}} R$. By changing ρ to $P^{-1}\rho P$, we may assume that $\rho(\alpha)$ and $\rho(\beta)$ generate the algebra $\mathcal{B}_2 \otimes_{\mathbb{Z}} R$. We can check that

$$\rho([\alpha, \beta]) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \tag{6}$$

where $u \in R^\times$. Put

$$\rho(\gamma) \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7}$$

Then we have

$$\begin{aligned}\rho([[\alpha, \beta], \gamma]) &= \frac{1}{ad - bc}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{acu + c^2u^2}{ad-bc} & u - \frac{a^2u + acu}{ad-bc} \\ \frac{c^2u}{ad-bc} & 1 - \frac{acu}{ad-bc} \end{pmatrix}.\end{aligned}$$

Since $\mathrm{tr}(\rho([[\alpha, \beta], \gamma])) = 2 + \frac{c^2u^2}{ad-bc} = 2$, we have $c^2 = 0$. By the hypothesis that R is reduced, we obtain $c = 0$, which implies that $\rho(\gamma) \in R[\rho(\alpha), \rho(\beta)]$.

Next we claim that if $(\rho(\gamma) \bmod I) \in (R/I)[\rho(\alpha), \rho(\beta)]$ for an ideal I of R with $I^2 = 0$, then we can show that $\rho(\gamma) \in R[\rho(\alpha), \rho(\beta)]$. This

claim and the result in the reduced case imply that $\rho(\gamma) \in R[\rho(\alpha), \rho(\beta)]$ for an arbitrary noetherian local ring R. As in the reduced case, we may assume that $\rho(\alpha)$ and $\rho(\beta)$ generate the algebra $\mathcal{B}_2 \otimes_{\mathbb{Z}} R$ and that (6) and (7) hold. Since $(\rho(\gamma) \bmod I) \in (R/I)[\rho(\alpha), \rho(\beta)]$, we have $c \in I$. By changing $\rho(\gamma)$ to $\rho(\gamma[\alpha, \beta])$, if necessary, we may assume that the $(1,2)$-entry b of $\rho(\gamma)$ is contained in $R^\times$. Remark that at least one of $\rho(\alpha), \rho(\beta)$ is not commutative with $\rho(\gamma)$ as elements of $\mathrm{GL}_2(R/m)$, where m is a maximal ideal of R. Let $\langle \alpha, \beta \rangle$ be the subgroup of Γ generated by α, β.

We see that there exists $\delta \in \langle \alpha, \beta \rangle$ such that $\rho(\delta)$ is not commutative with $\rho(\gamma)$ as elements of $\mathrm{GL}_2(R/m)$ and $\rho(\delta)$ has the form

$$\rho(\delta) = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$$

with $p - r \in R^\times$. Indeed, suppose that there exists no such δ. Then we have $\delta_1 \in \langle \alpha, \beta \rangle$ such that $\rho(\delta_1)$ is not commutative with $\rho(\gamma)$ as elements of $\mathrm{GL}_2(R/m)$ and $\rho(\delta_1)$ has the above form with $p - r \in m$. We also have $\delta_2 \in \langle \alpha, \beta \rangle$ such that $\rho(\delta_2)$ is commutative with $\rho(\gamma)$ as elements of $\mathrm{GL}_2(R/m)$ and $\rho(\delta_2)$ has the above form with $p - r \in R^\times$, because $\langle \rho(\alpha), \rho(\beta) \rangle$ generate $\mathcal{B}_2 \otimes_{\mathbb{Z}} R/m$. Putting $\delta = \delta_1 \delta_2$, we have such δ.

For such δ, we obtain

$$\mathrm{tr}(\rho([\gamma, \delta])) = 2 - c\frac{(p-r)\{b(p-r) + q(d-a)\}}{(ad-bc)pr}.$$

because $c \in I$ and $c^2 = 0$. Since $\rho(\gamma)$ and $\rho(\delta)$ are not commutative as elements of $\mathrm{GL}_2(R/m)$, $b(p-r) + q(d-a) \in R^\times$. From the assumption that $\mathrm{tr}(\rho([\gamma, \alpha])) = 2$, we have $c = 0$ because $p - r \in R^\times$. This implies that $\rho(\gamma) \in R[\rho(\alpha), \rho(\beta)]$. □

From the above proposition, we have the following corollary.

Corollary 4.2. *Let ρ be a representation of degree 2 for a group Γ on a scheme X. Suppose that ρ satisfies the conditions (i) and (ii) in Proposition 4.1. Then there exist an open covering $X = \cup_{i \in I} U_i$ and $P_i \in \mathrm{GL}_2(\Gamma(U_i, \mathcal{O}_X))$ such that*

$$P_i^{-1}(\rho|_{U_i})(\gamma)P_i = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

for $\gamma \in \Gamma$ and $i \in I$.

Remark 4.3. The condition (i) in Proposition 4.1 is necessary that ρ can be normalized into upper triangular matrices as in Corollary 4.2. Indeed, for such a representation ρ we always have $\mathrm{tr}(\rho([\alpha, \beta])) = 2$ for

$\alpha, \beta \in \Gamma$. Corollary 4.2 does not always hold without the condition (ii) in Proposition 4.1. The representation

$$\begin{array}{rccl} \rho & : & \mathbb{R} & \rightarrow & \mathrm{GL}_2(\mathbb{R}) \\ & & \theta & \mapsto & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{array}$$

satisfies the condition (i), but ρ has no nontrivial invariant subspace in $\mathbb{R}^2$. Indeed, the above representation ρ does not satisfy the condition (ii).

Remark 4.4. Note that any representation $\rho : \Gamma \rightarrow \mathrm{GL}_2(\mathbb{F}_2)$ over the field $\mathbb{F}_2$ for a group Γ is not a representation with Borel mold. Indeed, if there exists a representation ρ with Borel mold, then ρ can be normalized into upper triangular matrices. However the nonsingular upper triangular matrices over $\mathbb{F}_2$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and hence $\rho(\Gamma)$ is an abelian group. This is a contradiction.

From Theorem 3.2, we have a unique Γ-invariant sub-line bundle $\mathcal{L}_\Gamma$ of $\mathcal{O}^{\oplus 2}_{\mathrm{Rep}_2(\Gamma)_B}$ on $\mathrm{Rep}_2(\Gamma)_B$. We call $\mathcal{L}_\Gamma$ the *universal sub-line bundle* on $\mathrm{Rep}_2(\Gamma)_B$. Let us investigate the universal sub-line bundle $\mathcal{L}_\Gamma$.

Proposition 4.5. *Let* $\mathrm{F}_2 = \langle \alpha, \beta \rangle$ *be the free group of rank* 2*. The universal sub-line bundle* $\mathcal{L}_{\mathrm{F}_2}$ *over* $\mathrm{Rep}_2(\mathrm{F}_2)_B$ *is not trivial.*

Proof. We define the subvariety X of $\mathbb{C}^4$ by $X := \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1\}$. The group $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ acts on $\mathbb{C}^4$ and X by $(z_1, z_2, z_3, z_4) \mapsto (-z_1, -z_2, -z_3, -z_4)$. We define $\varphi : X \rightarrow \mathbb{C}^2 \setminus \{0\}$ by $(z_1, z_2, z_3, z_4) \mapsto (z_1 + \sqrt{-1}z_2, z_3 + \sqrt{-1}z_4)$. We denote by ψ the canonical projection $\mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{CP}^1$. The morphisms φ and ψ induce the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & \mathbb{C}^2 \setminus \{0\} & \xrightarrow{\psi} & \mathbb{CP}^1 \\ \downarrow & & \downarrow & & \| \\ X/\{\pm 1\} & \xrightarrow{\overline{\varphi}} & (\mathbb{C}^2 \setminus \{0\})/\{\pm 1\} & \xrightarrow{\overline{\psi}} & \mathbb{CP}^1. \end{array}$$

We denote by f and $\overline{f}$ the compositions $\psi \circ \varphi$ and $\overline{\psi} \circ \overline{\varphi}$, respectively. Let us consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^1}(-1) \rightarrow \mathcal{O}^{\oplus 2}_{\mathbb{CP}^1} \rightarrow \mathcal{O}_{\mathbb{CP}^1}(1) \rightarrow 0$$

over $\mathbb{CP}^1$. Taking the pull back by $\overline{f}$, we have the following exact sequence

$$0 \to \overline{f}^* \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathcal{O}_{X/\{\pm 1\}}^{\oplus 2} \to \overline{f}^* \mathcal{O}_{\mathbb{CP}^1}(1) \to 0. \tag{8}$$

We put $\mathcal{L} := \overline{f}^* \mathcal{O}_{\mathbb{CP}^1}(-1)$ and $\mathcal{M} := \overline{f}^* \mathcal{O}_{\mathbb{CP}^1}(1)$. Note that $\mathcal{L} \cong \mathcal{M}^{-1}$. We easily see that $\overline{\psi}^* \mathcal{O}_{\mathbb{CP}^1}(-1)^{\otimes 2}$ is trivial on $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$, and hence we have $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_{X/\{\pm 1\}}$ and $\mathcal{M} \cong \mathcal{L}$. The varieties X and $X/\{\pm 1\}$ has the same homotopy types as S^3 and $\mathbb{RP}^3$, respectively. We can regard the above diagram as follows up to homotopy:

$$\begin{array}{ccc} S^3 & \xrightarrow{f} & S^2 \\ \pi \downarrow & & \| \\ \mathbb{RP}^3 & \xrightarrow{\overline{f}} & S^2. \end{array}$$

Here the map π is the canonical projection. The map f is the Hopf map. The first Chern class $c_1(\mathcal{O}_{\mathbb{CP}^1}(-1))$ is a generator of $\mathrm{H}^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$ and $c_1(\mathcal{L}) \neq 0$ in $\mathrm{H}^2(\mathbb{RP}^3, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $\mathcal{L}$ is not trivial as a topological vector bundle. Therefore we see that $\mathcal{L} \ncong \mathcal{O}_{X/\{\pm 1\}}$.

Let us define a representation with Borel mold on $X/\{\pm 1\}$. On the affine variety $X/\{\pm 1\}$, the exact sequence (8) splits, and hence we have $\mathcal{O}_{X/\{\pm 1\}}^{\oplus 2} \cong \mathcal{L} \oplus \mathcal{M} \cong \mathcal{L} \oplus \mathcal{L}$. We put $R := \Gamma(X/\{\pm 1\}, \mathcal{O}_{X/\{\pm 1\}})$. Through the identification $\mathrm{M}_2(R) = \mathrm{End}_{\mathcal{O}_{X/\{\pm 1\}}}(\mathcal{O}_{X/\{\pm 1\}}^{\oplus 2}) = \mathrm{End}_{\mathcal{O}_{X/\{\pm 1\}}}$ $(\mathcal{L} \oplus \mathcal{M})$, we define $\rho : \mathrm{F}_2 \to \mathrm{GL}_2(R)$ by

$$\rho(\alpha) := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \rho(\beta) := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$
$$\in \begin{pmatrix} \mathrm{Hom}(\mathcal{L}, \mathcal{L}) & \mathrm{Hom}(\mathcal{M}, \mathcal{L}) \\ \mathrm{Hom}(\mathcal{L}, \mathcal{M}) & \mathrm{Hom}(\mathcal{M}, \mathcal{M}) \end{pmatrix}.$$

Note that $\mathrm{Hom}(\mathcal{L}, \mathcal{L}) = \mathrm{Hom}(\mathcal{L}, \mathcal{M}) = \cdots = \mathcal{O}_{X/\{\pm 1\}}$. We see that ρ is a representation with Borel mold and that $\mathcal{L}$ is a unique Γ-invariant sub-line bundle. For the morphism $g : X/\{\pm 1\} \to \mathrm{Rep}_2(\mathrm{F}_2)_B$ associated to ρ, we have $g^* \mathcal{L}_{\mathrm{F}_2} = \mathcal{L} \ncong \mathcal{O}_{X/\{\pm 1\}}$. This implies that $\mathcal{L}_{\mathrm{F}_2}$ is not trivial. □

Corollary 4.6. *On the representation variety of degree* 2 *with Borel mold* $\mathrm{Rep}_2(2)_B$ *for the free monoid of rank* 2*, the universal sub-line bundle* $\mathcal{L}_2$ *is not trivial.*

Proof. Let $\Upsilon_2 = \langle \alpha, \beta \rangle$ be the free monoid of rank 2. The inclusion $\Upsilon_2 \to \mathrm{F}_2$ ($\alpha \mapsto \alpha$, $\beta \mapsto \beta$) induces the morphism $\phi : \mathrm{Rep}_2(\mathrm{F}_2)_B \to \mathrm{Rep}_2(2)_B$. We easily see that $\phi^* \mathcal{L}_2 = \mathcal{L}_{\mathrm{F}_2}$. By the previous proposition the universal sub-line bundle $\mathcal{L}_{\mathrm{F}_2}$ is not trivial, neither is $\mathcal{L}_2$. □

We show that the universal sub-line bundle $\mathcal{L}_2$ is a 2-torsion element of the Picard group.

Proposition 4.7. *For the universal sub-line bundle $\mathcal{L}_2$ on $\mathrm{Rep}_2(2)_B$, we have $\mathcal{L}_2^{\otimes 2} \cong \mathcal{O}_{\mathrm{Rep}_2(2)_B}$.*

Proof. We denote $\mathrm{B}_2(\Upsilon_2)_B$ by $\mathrm{B}_2(2)_B$ for the free monoid Υ_2. Let us consider the morphism

$$\begin{array}{cccc} \psi_2 & : & \mathrm{B}_2(2)_B \times \mathrm{GL}_2 & \to & \mathrm{Rep}_2(2)_B \\ & & (\rho, P) & \mapsto & P^{-1}\rho P. \end{array}$$

We can easily see that ψ_2 is a smooth morphism. (Furthermore, ψ_2 is a principal fiber bundle with fiber $\tilde{B}_2 := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_2$.) We denote the universal nonsingular 2×2 matrix on GL_2 by $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. We also denote the universal representations in $\mathrm{B}_2(2)_B$ and $\mathrm{Rep}_2(2)_B$ by

$$\rho_{\mathrm{B}}(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ 0 & d(\gamma) \end{pmatrix}$$

and

$$\rho_{\mathrm{Rep}}(\gamma) = \begin{pmatrix} a'(\gamma) & b'(\gamma) \\ c'(\gamma) & d'(\gamma) \end{pmatrix}$$

for $\gamma \in \Upsilon_2 = \langle \alpha, \beta \rangle$, respectively. By putting $u := (a(\alpha) - d(\alpha))b(\beta) - b(\alpha)(a(\beta) - d(\beta))$, we have

$$\rho_{\mathrm{B}}(\alpha)\rho_{\mathrm{B}}(\beta) - \rho_{\mathrm{B}}(\beta)\rho_{\mathrm{B}}(\alpha) = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}.$$

Since ρ_{B} is a representation with Borel mold, u is an invertible global function on $\mathrm{B}_2(2)_B$. Hence $\mathrm{Ker}(\rho_{\mathrm{B}}(\alpha)\rho_{\mathrm{B}}(\beta) - \rho_{\mathrm{B}}(\beta)\rho_{\mathrm{B}}(\alpha))$ is the universal sub-line bundle on $\mathrm{B}_2(2)_{\mathrm{B}}$. Through the morphism ψ_2 we have

$$\begin{aligned} &\rho_{\mathrm{Rep}}(\alpha)\rho_{\mathrm{Rep}}(\beta) - \rho_{\mathrm{Rep}}(\beta)\rho_{\mathrm{Rep}}(\alpha) \\ &\quad = \begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} (\rho_{\mathrm{B}}(\alpha)\rho_{\mathrm{B}}(\beta) - \rho_{\mathrm{B}}(\beta)\rho_{\mathrm{B}}(\alpha)) \begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ &\quad = \frac{1}{\Delta} \begin{pmatrix} rsu & s^2u \\ -r^2u & -rsu \end{pmatrix}, \end{aligned}$$

where $\Delta := ps - qr$. Since $\mathrm{Ker}(\rho_{\mathrm{Rep}}(\alpha)\rho_{\mathrm{Rep}}(\beta) - \rho_{\mathrm{Rep}}(\beta)\rho_{\mathrm{Rep}}(\alpha))$ is equal to the universal sub-line bundle $\mathcal{L}_2 \subseteq \mathcal{O}_{\mathrm{Rep}_2(2)_B}^{\oplus 2}$, ${}^t(-s^2u/\Delta, rsu/\Delta)$ and ${}^t(rsu/\Delta, -r^2u/\Delta)$ are global sections of $\mathcal{L}_2$. We define the prime divisors D_1 and D_2 on $\mathrm{Rep}_2(2)_B$ by $D_1 := \psi_2(\{r = 0\})$ and $D_2 :=$

$\psi_2(\{s = 0\})$, respectively. Let us denote the $(1,2)$-entry and $(2,1)$-entry of $\rho_{\mathrm{Rep}}(\alpha)\rho_{\mathrm{Rep}}(\beta) - \rho_{\mathrm{Rep}}(\beta)\rho_{\mathrm{Rep}}(\alpha)$ by b' and c', respectively. Then $D_1 = \{c' = 0\}$ and $D_2 = \{b' = 0\}$ set-theorically. From the two global sections above we see that $\mathcal{L}_2 \sim D_1 \sim D_2$. Because $2 \cdot D_1 \sim \operatorname{div}(c') \sim 0$, we conclude that $\mathcal{L}_2^{\otimes 2} \cong \mathcal{O}_{\mathrm{Rep}_2(2)_B}$. □

Corollary 4.8. *Let $\mathcal{L}_{\mathrm{F}_2}$ be the universal sub-line bundle on $\mathrm{Rep}_2(\mathrm{F}_2)_{\mathrm{B}}$. Then $\mathcal{L}_{\mathrm{F}_2}^{\otimes 2} \cong \mathcal{O}_{\mathrm{Rep}_2(\mathrm{F}_2)_{\mathrm{B}}}$.*

Proof. The statement follows from that the pull-back of $\mathcal{L}_2$ by the morphism $\mathrm{Rep}_2(\mathrm{F}_2)_{\mathrm{B}} \to \mathrm{Rep}_2(2)_{\mathrm{B}}$ is equal to $\mathcal{L}_{\mathrm{F}_2}$. □

From Corollary 4.8 we have the following:

Corollary 4.9. *Let Γ be a group generated by two elements. Let R be a commutative ring such that $\mathrm{Pic}(\mathrm{Spec}(R))$ has no 2-torsion element. For each representation $\rho : \Gamma \to \mathrm{GL}_2(R)$ with Borel mold, we have some $P \in \mathrm{GL}_2(R)$ such that*

$$P\rho(\gamma)P^{-1} = \begin{pmatrix} a(\gamma) & b(\gamma) \\ 0 & d(\gamma) \end{pmatrix}$$

for each $\gamma \in \Gamma$.

Proof. Let us consider a closed immersion $f : \mathrm{Rep}_2(\Gamma)_{\mathrm{B}} \to \mathrm{Rep}_2(\mathrm{F}_2)_{\mathrm{B}}$ induced by a surjective morphism $\mathrm{F}_2 \to \Gamma$. From Corollary 4.8, we see that $f^*\mathcal{L}_{\mathrm{F}_2} = \mathcal{L}_\Gamma$ is a 2-torsion element of the Picard group. Hence the pull-back of $\mathcal{L}_\Gamma$ on R is trivial, which follows the claim. □

Let us discuss the free group of rank ≥ 3 case.

Proposition 4.10. *For the free group F_m with $m \geq 3$, we have $\mathcal{L}_{\mathrm{F}_m}^{\otimes n} \not\cong \mathcal{O}_{\mathrm{Rep}_2(\mathrm{F}_m)_B}$ for each integer $n \neq 0$.*

Proof. There exists an affine scheme X (over $\mathbb{C}$) which satisfies the following condition: X has a sub-line bundle $\mathcal{L} \subseteq \mathcal{O}_X^{\oplus 2}$ such that $\mathcal{L}^{\otimes n} \not\cong \mathcal{O}_X$ for each integer $n \neq 0$. Example 4.12 gives us such an affine scheme X. We denote $\mathcal{O}_X^{\oplus 2}/\mathcal{L}$ by $\mathcal{M}$. Then we have $\mathcal{O}_X^{\oplus 2} \cong \mathcal{L} \oplus \mathcal{M}$ and $\mathcal{M} \cong \mathcal{L}^{-1}$. Since $\mathcal{L}$ is generated by two global sections, $\mathrm{Hom}(\mathcal{M}, \mathcal{L}) \cong \mathcal{L}^{\otimes 2}$ is generated by some two global sections. Suppose that $s, t \in \mathrm{Hom}(\mathcal{M}, \mathcal{L})$ are global sections which generates $\mathrm{Hom}(\mathcal{M}, \mathcal{L})$. For $\mathrm{F}_3 = \langle \alpha, \beta, \gamma \rangle$, we define a representation ρ_3 on X by

$$\rho_3(\alpha) := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \rho_3(\beta) := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \rho_3(\gamma) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\in \mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus 2}) = \begin{pmatrix} \mathrm{Hom}(\mathcal{L},\mathcal{L}) & \mathrm{Hom}(\mathcal{M},\mathcal{L}) \\ \mathrm{Hom}(\mathcal{L},\mathcal{M}) & \mathrm{Hom}(\mathcal{M},\mathcal{M}) \end{pmatrix}.$$

For the free group F_m with $m \geq 4$, we define a representation ρ_m on X by taking the composite of a surjection $\mathrm{F}_m \twoheadrightarrow \mathrm{F}_3$ and the above ρ_3. We can check that ρ_m is a representation with Borel mold. For the morphism $g : X \to \mathrm{Rep}_2(\mathrm{F}_m)_B$ associated to ρ_m, we have $g^*\mathcal{L}_{\mathrm{F}_m} \cong \mathcal{L}$. This implies that $\mathcal{L}_{\mathrm{F}_m}^{\otimes n} \ncong \mathcal{O}_{\mathrm{Rep}_2(\mathrm{F}_m)_B}$ for each integer $n \neq 0$. □

Corollary 4.11. *For the universal sub-line bundle $\mathcal{L}_m$ for the free monoid Υ_m with $m \geq 3$, we have $\mathcal{L}_m^{\otimes n} \ncong \mathcal{O}_{\mathrm{Rep}_2(m)_B}$ for each integer $n \neq 0$.*

Proof. We can prove the statement in the same way as Corollary 4.6. □

The following example has been used in the proof of the previous proposition.

Example 4.12. Let us consider the affine variety $X := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^2 + z_2^2 + z_3^2 = 1\}$. The variety X has the same homotopy type as S^2. We define the morphism $f : X \to \mathbb{CP}^1$ by $(z_1, z_2, z_3) \mapsto (z_1 + \sqrt{-1}z_2, z_3)$. Then the morphism f can be regarded as a degree 2 map $S^2 \to S^2$ up to homotopy. We define the sub-line bundle $\mathcal{L}$ on X by $\mathcal{L} := f^*\mathcal{O}_{\mathbb{CP}^1}(-1) \subset \mathcal{O}_X^{\oplus 2}$. Since $c_1(\mathcal{L}) \neq 0 \in \mathrm{H}^2(X, \mathbb{Z}) \cong \mathbb{Z}$, the line bundle $\mathcal{L}$ is not a torsion in the Picard group $\mathrm{Pic}(X)$.

Remark 4.13. From Propositions 4.5 and 4.7 we see that the 2-torsion part of the cohomology does not vanish: $\mathrm{H}^2(\mathrm{Rep}_2(\mathrm{F}_2)_B \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{Z})_2 \neq 0$. By Proposition 4.10 we also see that the free part of $\mathrm{H}^2(\mathrm{Rep}_2(\mathrm{F}_m)_B \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{Z})$ does not vanish for each $m \geq 3$. The topology of the representation varieties and the character varieties with Borel mold has been investigated in [6], [7], and [8].

5. Appendix

In this appendix, we prove several propositions on discriminants in the degree 2 case. These propositions have been used in the previous section. Discriminants are invariants which describe open subschemes of absolutely irreducible representations in the representation varieties. More precisely, see [9], [4], and [5].

Definition 5.1 ([9])**.** Let R be a commutative ring.
For $A, B \in \mathrm{M}_2(R)$ we define the *discriminant* $\Delta(A, B)$ by

$$\begin{aligned} \Delta(A,B) \ := \ & \mathrm{tr}(A)^2 \det(B) + \mathrm{tr}(B)^2 \det(A) + \mathrm{tr}(AB)^2 \\ & -\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) - 4\det(A)\det(B). \end{aligned}$$

From the definition we see that $\Delta(A,B)=\Delta(B,A)$.

Remark 5.2. The discriminant $\Delta(A,B)$ above is closely related to the discriminant in [4]. For $A,B,C,D\in \mathrm{M}_2(R)$ we define the *discriminant* of degree 2 in [4] by

$$\Delta(A,B,C,D):=\det\begin{pmatrix} a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\\ c_1 & c_2 & c_3 & c_4\\ d_1 & d_2 & d_3 & d_4\end{pmatrix},$$

where

$$A=\begin{pmatrix} a_1 & a_2\\ a_3 & a_4\end{pmatrix}, B=\begin{pmatrix} b_1 & b_2\\ b_3 & b_4\end{pmatrix}, C=\begin{pmatrix} c_1 & c_2\\ c_3 & c_4\end{pmatrix}, D=\begin{pmatrix} d_1 & d_2\\ d_3 & d_4\end{pmatrix}.$$

Note that $\Delta(A,B,C,D)\in R^\times$ if and only if $\{A,B,C,D\}$ is an R-basis of $\mathrm{M}_2(R)$. For $A,B\in \mathrm{M}_2(R)$, we have $\Delta(A,B)=-\Delta(I_2,A,B,AB)$.

We can easily obtain the following proposition.

Proposition 5.3 (cf. [9])**.** *For each $A,B\in \mathrm{GL}_2(R)$, we have*

$$\Delta(A,B)=\det(AB)(\mathrm{tr}(ABA^{-1}B^{-1})-2).$$

The following proposition has been used in Proposition 4.1.

Proposition 5.4. *Let $A,B\in \mathrm{M}_2(R)$. Assume that $\Delta(A,B)=0$. Then the R-subalgebra $R[A,B]$ of $\mathrm{M}_2(R)$ generated by A and B is contained in the R-submodule $R\cdot I_2+R\cdot A+R\cdot B$.*

Proof. First we show the claim that AB is expressed as a linear combination of $\{I_2,A,B\}$. For proving this, we can assume that R is a local ring. Indeed, let us define the ideal J of R by

$$J:=\{a\in R\mid aAB \text{ is expressed as a linear combination of } I_2,A,B\ \}.$$

If the claim is true for the local ring case, then AB is expressed as a linear combination of I_2,A,B in $\mathrm{M}_2(R_\wp)$ for each prime ideal $\wp\in \mathrm{Spec}R$. Hence $J\not\subset\wp$ for each $\wp\in\mathrm{Spec}R$, which implies $J=R$. Since $1\in J$, the claim is true for an arbitrary ring R.

Assume that R is a local ring. Since $\Delta(A,B)=\Delta(I_2,A,B,AB)=0$, one of $\{I_2,A,B,AB\}$ is expressed as a linear combination of the other three elements by Remark 5.2. If the one is AB, then the claim is obvious. Suppose that $A=c_1I_2+c_2B+c_3AB$. If $c_3\in R^\times$, then

$AB = c_3^{-1}(A - c_1 I_2 - c_2 B)$. If $c_3 \notin R^\times$, then by multiplying B from the right we have

$$
\begin{aligned}
AB &= c_1 B + c_2 B^2 + c_3 AB^2 \\
&= c_1 B + c_2(\mathrm{tr}(B)B - \det(B)I_2) + c_3 A(\mathrm{tr}(B)B - \det(B)I_2) \\
&= (c_1 + c_2\mathrm{tr}(B))B - c_2 \det(B)I_2 - c_3 \det(B)A + c_3\mathrm{tr}(B)AB \\
&= -(c_2 + c_1 c_3)\det(B)I_2 + (c_1 + c_2\mathrm{tr}(B) - c_2 c_3 \det(B))B \\
&\quad + c_3(\mathrm{tr}(B) - c_3 \det(B))AB.
\end{aligned}
$$

Hence $(1 - c_3(\mathrm{tr}(B) - c_3 \det(B)))AB$ is contained in $R \cdot I_2 + R \cdot B$. Since $1 - c_3(\mathrm{tr}(B) - c_3 \det(B)) \in R^\times$, AB can be expressed as a linear combination of I_2 and B. In the case that $B = c_1 I_2 + c_2 A + c_3 AB$, we can also prove that $AB \in R \cdot I_2 + R \cdot A + R \cdot B$ in the same way. If $I_2 = c_1 A + c_2 B + c_3 AB$, then at least one of c_i is contained in $R^\times$, and hence the claim can be easily checked. Thus AB is expressed as a linear combination of $\{I_2, A, B\}$. Similarly we can prove that BA is expressed as a linear combination of $\{I_2, A, B\}$.

Next we show that any monomial of A and B is contained in the R-submodule $R \cdot I_2 + R \cdot A + R \cdot B$. Each monomial of length ≥ 2 contains one of AA, AB, BA, and BB as a subsequence. By the Cayley-Hamilton theorem and the above discussion, we can reduce AA, AB, BA, and BB to monomials of length one. Hence each monomial is contained in $R \cdot I_2 + R \cdot A + R \cdot B$ by induction. Thus we have completed the proof. $\square$

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